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WOLD DECOMPOSITION FOR OPERATORS CLOSE ISOMETRIES

Laura MANOLESCU

Abstract

The Wold decomposition theorem is used to decompose a wide sense stationary random process into its deterministic and purely nondeterministic parts and, also, has an important role in signal processing, factorization theory and the description of dilations in Hilbert spaces. In geometric language, this means that an isometry is a direct sum between an unitary operator and a shift.

In Bergman and Dirichlet spaces, the shift operator is not an isometry, but it is a left invertible operator. In this paper we give conditions on the left invertible operators such that a operator version, in the sense of Rosenblum and Rovnyak, of the Wold decomposition to take place.[1]

1 Introduction

The Wold decomposition theorem [18] applies to the analysis of stationary random processes. It provides a representation of such processes and also an interpretation of the representation in terms of linear prediction: an arbitrary unpredictable process can be written as an orthogonal sum of a regular process and a predictable process [18].

In 1961, Paul R. Halmos [9] gave the following form of the Wold decomposition theorem in operator language:

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Keywords and phrases: *Wold decomposition, wandering subspace,left invertible operators, random processes, frames in Hilbert spaces*

Theorem 1.1. *Let V be an isometry on a Hilbert space $\mathcal{H}$. Then there is a decomposition of $\mathcal{H}$ as a direct sum of two mutually orthogonal subspaces*

$$\mathcal{H} = \mathcal{H}_\infty \bigoplus \mathcal{H}_s$$

such that

(i) $\mathcal{H}_\infty$ *and* $\mathcal{H}_s$ *reduce* V.

(ii) The restriction of V *to* $\mathcal{H}_\infty$ *is an unitary operator.*

(iii) The restriction of V *to* $\mathcal{H}_s$ *is unitarily equivalent to an unilateral shift.*

The decomposition is unique.

We recall that a subspace $\mathcal{H}_0$ of $\mathcal{H}$ reduce V if $\mathcal{H}_0$ is invariant to V and its adjoint. In fact, the subspaces $\mathcal{H}_\infty$ and $\mathcal{H}_s$ are obtained in the following manner

$$\mathcal{H}_\infty = \bigcap_{n=1}^{\infty} V^n \mathcal{H}$$

$$\mathcal{H}_s = \bigoplus_{n=1}^{\infty} V^n W,$$

where $W := \mathcal{H} \ominus V\mathcal{H}$ is the orthogonal complement of $V\mathcal{H}$ in $\mathcal{H}$. Here, W is a wandering subspace for V, that is

$$V^m W \perp V^n W, \quad m \neq n.$$

See also [10], [14].

Theorem 1.1 has some remarkable consequences (see [3], [11]) such as:

- the deduction of the Beurling's invariant subspace Theorem in Hardy spaces;
- the description of the structure of a wide-sense stationary random sequence;
- the description of the structure of isometric and unitary dilation spaces for contractions of a Hilbert space (see Nagy and Foiaş [17]).

The above version of the Wold decomposition emphasizes spatial structure. An operator version of the Wold decomposition of an isometry is given by M. Rosenblum and J. Rovyak in their book [13].

But in Bergman and Dirichlet spaces the shift operator is no longer an isometry. Remarkable Wold type Theorems for classes of left invertible operators and applications to problems of invariant subspaces were obtained by S. Richter [12] and S. Shimorin [16]. In 1991, A. Aleman, S. Richter, C. Sundberg [2] proved the Beurling type theorem for Bergman shift, which was a big step in the study of invariant subspaces of the Bergman shift. This result became an important tool in the function theory of L_a^2 because it shows the structure of invariant subspaces of the Bergman space.

This paper is motivated by a problem posed by S. Shimorin [16]. The problem is to give new conditions for a left invertible operator to imply Wold type decompositions. The aim of this paper is to give conditions on the left invertible operators such that a operator version of the Wold decomposition can be proved. The left invertible operators (the operators bounded below) are the analysis operators from frame theory (see [5], [8]).

2 Results

We denote by $\mathcal{L}(\mathcal{H})$ the algebra of all linear bounded operators on the Hilbert space $\mathcal{H}$ and for $T \in \mathcal{L}(\mathcal{H})$, we denote by T^* the adjoint operator of T. The following lemma is a well-known result.

Lemma 2.1. *Let be $T \in \mathcal{L}(\mathcal{H})$. The following are equivalent*

(i) T *is left invertible;*

(ii) T *is bounded below, i.e. there exists a constant* $m > 0$ *such that*

$$\|Th\| \geq m\|h\|, \quad h \in \mathcal{H};$$

(iii) T^* *is surjective;*

(iv) T^*T *is invertible.*

If T is left invertible, then $T\mathcal{H}$ is a closed subspace of $\mathcal{H}$. As in [12, 16] we distinguish the following left-inverse of T

$$T^- = (T^*T)^{-1}T^*$$

and its kernel

$$W = \mathcal{H} \ominus T\mathcal{H} = KerT^*.$$

The subspace W is called the defect of T. It is clear that if T is left invertible then T^n is left invertible.

In the following, $\mathcal{D}$ denotes the set of all left invertible operators on $\mathcal{H}$ for which the following condition holds

$$(T^n)^- = (T^-)^n, \quad \text{for all } n \geq 2. \tag{1}$$

Remark 2.2. *If T is an isometry, then $T^*T = I$, hence $T^- = T^*$ and (1) holds. In fact, if T is left invertible and $T^- = T^*$, then T is an isometry. Indeed, from the relation $(T^*T)^{-1}T^* = T^*$ it follows*

$$(T^*T)^{-1}T^*T = T^*T$$

hence $I = T^*T$.

In the following we give conditions for operators to be in the class $\mathcal{D}$.

Proposition 2.3. *Let $Q \in \mathcal{L}(\mathcal{H})$ be left invertible and quasinormal. Then $Q \in \mathcal{D}$.*

Proof. Since Q quasinormal, we have $(Q^*Q)Q = Q(Q^*Q)$.

We claim that this implies that $Q^{*n}Q^n = (Q^*Q)^n$, for all $n \geq 2$. We prove this by induction on n.

Indeed, for $n = 2$

$$Q^{*2}Q^2 = Q^*(Q^*Q)Q = (Q^*Q)(Q^*Q)$$

If $Q^{*n}Q^n = (Q^*Q)^n$ then

$$\begin{aligned}(Q^*Q)^{n+1} &= Q^*Q(Q^*Q)^n = (Q^*Q)Q^{*n}Q^n \\ &= Q^{*n}(Q^*Q)Q^n.\end{aligned}$$

We used the fact that Q is quasinormal, hence $(Q^*Q)Q^* = Q^*(Q^*Q)$ and by induction, we have $(Q^*Q)Q^{*n} = Q^{*n}(Q^*Q)$.

So we get$(Q^*Q)^{n+1} = Q^{*n+1}Q^{n+1}$.

It follows

$$(Q^*Q)^{-n} = (Q^{*n}Q^n)^{-1}.$$

Hence

$$\begin{aligned}(Q^-)^n &= [(Q^*Q)^{-1}Q^*]^n = Q^{*n}(Q^*Q)^{-n} \\ &= Q^{*n}(Q^{*n}Q^n)^{-1} = (Q^n)^-.\end{aligned}$$

□

We recall that two operators T_1, T_2 are *double commuting* if $T_1T_2{=}T_2T_1$ and $T_1T_2^*{=}T_2^*T_1$

Proposition 2.4. *Let T_1, T_2 be double commuting operators in $\mathcal{D}$. Then*

$$T_1T_2 \in \mathcal{D}.$$

Proof. It is clear that T_1T_2 is left invertible since

$$T_2^-T_1^-T_1T_2 = T_2^-T_2 = I.$$

From hypothesis,

$$T_2(T_1^*T_1) = (T_1^*T_1)T_2$$

hence

$$(T_1^*T_1)^{-1}T_2 = T_2(T_1^*T_1)^{-1}.$$

It follows that

$$\begin{aligned} T_1T_2 &= [(T_1T_2)^*(T_1T_2)]^{-1}(T_1T_2)^* \\ &= (T_2^*T_2T_1^*T_1)^{-1}T_2^*T_1^* \\ &= (T_1^*T_1)^{-1}(T_2^*T_2)^{-1}T_1^*T_2^* \\ &= (T_1^*T_1)^{-1}T_1^*(T_2^*T_2)^{-1}T_2^* \\ &= T_1^-T_2^-. \end{aligned}$$

From here we get

$$\begin{aligned} [(T_1T_2)^-]^n &= (T_1^-T_2^-)^n = (T_1^-)^n(T_2^-)^n \\ &= (T_1^n)^-(T_2^n)^- = (T_1^nT_2^n)^- = [(T_1T_2)^n]^-, \end{aligned}$$

since $T_1^n, T_2^n \in \mathcal{D}$ and T_1^n and T_2^n are double commuting. □

Corollary 2.5. *Let be $T_1 \in \mathcal{D}$ and T_2 be normal and invertible, and $T_1T_2 = T_2T_1$. Then $T_1T_2 \in \mathcal{D}$.*

Proof. From the hypothesis it follows that $T_2 \in \mathcal{D}$. From the Fuglede-Putnam theorem [10] it follows $T_1^*T_2 = T_2T_1^*$. Now, the conclusion is a consequence of Proposition 2.4. □

Remark 2.6. *From the above corrollary, we also obtain that every quasinormal and left invertible operator is in $\mathcal{D}$.*

Indeed, T admits polar decomposition $T = VA$, with V isometry,
$A = (T^*T)^{1/2}$ and $VA = AV$ (see [7]).

Next, we give an example of quasinormal, left invertible operator that is not an isometry.

Let $\mathcal{K}$ be a Hilbert space of dimension at least 2 and

$$l^2(\mathcal{K}) = \{\widetilde{k} = (k_0, k_1, \ldots, k_n, \ldots) | k_j \in \mathcal{K}, j = 0, 1, \ldots\} \text{ and } \|\widetilde{k}\|_2^2 := \sum_{j=0}^{\infty} \|k_j\|^2 < \infty\}.$$

Let $L \in \mathcal{L}(\mathcal{K})$ be a positive invertible operator such that $\|Lk\| \geq m\|k\|$, for all k and some $m > 1$. We define the following operator on $l^2(\mathcal{K})$:

$$T\widetilde{k} := (0, Lk_0, Lk_1, \ldots)$$

Note that T is bounded on $l^2(\mathcal{K})$ and is not surjective;

$$(T^*T\widetilde{k})_n = L^2 k_n$$

$$[T(T^*T)\widetilde{k}] = (0, L^3 k_0, \ldots, L^3 k_n, \ldots)$$

$$(T^*T)T\widetilde{k} = T^*T(0, Lk_0, \ldots, Lk_n, \ldots) = (0, L^3 k_0, \ldots, L^3 k_n, \ldots)$$

It follows that T is quasinormal.
We have $\|T\widetilde{k}\|_2^2 = \sum \|Lk_n\|^2 \geq m^2 \sum \|k_n\|^2$.
This implies that T is not an isometry, since $\|T\widetilde{k}\|_2^2 \neq \sum \|k_n\|^2$.

We give conditions for the weighted shifts [15] and weighted translation operators [6] to be in $\mathcal{D}$.

Proposition 2.7. *Every bounded left invertible unilateral weighted shift on l^2 is in the class $\mathcal{D}$.*

Proof. Let T be a unilateral weighted shift, which is bounded below, ie
$Te_k = w_k e_{k+1}, \quad k \geq 0$ and $C_1 \leq w_n \leq C_2$, where C_1, C_2 are positive constants.

We have:

$$T^* e_k = \begin{cases} \overline{w}_{k-1} e_{k-1}, & k \geq 1 \\ 0, & k = 0; \end{cases}$$

$$T^n e_k = w_k w_{k+1} \cdots w_{k+n-1} e_{k+n}, \text{ and}$$

$$T^{*n} e_k = \begin{cases} \overline{w}_{k-1} \overline{w}_{k-2} \cdots \overline{w}_{k-n} e_{k-n}, & k \geq n \\ 0, & 0 \leq k < n. \end{cases}$$

It follows

$$\begin{aligned} (T^-)^* e_k = T(T^*T)^{-1} e_k &= T\left(\frac{1}{|w_k|^2} e_k\right) \\ &= \frac{1}{|w_k|^2} w_k e_{k+1} \\ &= \frac{1}{\overline{w}_k} e_{k+1} \end{aligned}$$

and

$$(T^-)^{*n} e_k = \frac{1}{\overline{w}_k \overline{w}_{k+1} \cdots \overline{w}_{k+n-1}} e_{k+n}.$$

On the other hand,

$$T^{*n} T^n e_k = w_k w_{k+1} \cdots w_{k+n-1} \overline{w}_{k+n-1} \cdots \overline{w}_k e_k$$

which implies

$$\begin{aligned} [(T^n)^-]^* &= T^n \left(\frac{1}{|w_k|^2 \cdots |w_{k+n-1}|^2} \right) e_k \\ &= \frac{1}{\overline{w}_k \cdots \overline{w}_{k+n-1}} e_{k+n} \end{aligned}$$

It follows $(T^-)^n = (T^n)^-$, for all $n \geq 2$. □

From the above Proposition, it follows that the Bergman shifts, i.e. the shifts with sequence weights $\left\{ \sqrt{\frac{k+1}{k+2}} \right\}_{k \in \mathbb{N}}$ and also, the Dirichlet shifts, i.e. the shifts with sequence weights $\left\{ \sqrt{\frac{k+2}{k+1}} \right\}_{k \in \mathbb{N}}$ are in the class of $\mathcal{D}$.

Proposition 2.8. *Every left invertible weighted translation operator on $L^2(0, \infty)$ is in $\mathcal{D}$.*

Proof. Let T be a weighted translation operator, i.e.

$$Tf(x) = \begin{cases} \dfrac{\varphi(x)}{\varphi(x-t)} f(x-t), & \text{if } x > t \\ 0, & \text{if } 0 < x \leq t \end{cases}$$

Further we have

$$T^* f(x) = \frac{\varphi(x+t)}{\varphi(x)} f(x+t)$$

$$T^n f(x) = \frac{\varphi(x)}{\varphi(x-nt)} f(x-nt), \text{ for } x > nt$$

and

$$T^{*n} f(x) = \frac{\varphi(x+nt)}{\varphi(x)} f(x+nt).$$

It follows that

$$T^{*n} T^n f(x) = \left[\frac{\varphi(x+nt)}{\varphi(x)} \right]^2 f(x),$$

$$T^*Tf(x) = \left[\frac{\varphi(x+t)}{\varphi(x)}\right]^2 f(x),$$

$$T^- f(x) = \frac{\varphi(x)}{\varphi(x+t)} f(x+t),$$

$$(T^-)^n f(x) = \frac{\varphi(x)}{\varphi(x+nt)} f(x+nt) \text{ and}$$

$$(T^n)^- f(x) = \frac{\varphi(x)}{\varphi(x+nt)} f(x+nt).$$

□

Next, we give the main result of this paper. We recall here the following notation $T^- = (T^*T)^{-1}T^*$.

Theorem 2.9. *Let $T \in \mathcal{L}(\mathcal{H})$ be in $\mathcal{D}$. Then*

(i) $P_0 := I - TT^-$ *is the projection of $\mathcal{H}$ on $W := \mathcal{H} \ominus T\mathcal{H}$;*

(ii) *as $n \to \infty$, $T^n(T^-)^n$ converges strongly to the projection operator, P, on*

$$\bigcap_{n=1}^{\infty} T^n\mathcal{H};$$

(iii) $\sum_{j=0}^{\infty} T^j P_0 (T^-)^j$ *converges strongly to* $Q := I - P$;

(iv) $Q\mathcal{H} = \{h \in \mathcal{H} : \lim_{n\to\infty} \|(T^{*n}T^n)^{-1/2}T^{*n}\| = 0\}$;

(v) $P\mathcal{H}$ *and* $Q\mathcal{H}$ *reduce* T;

(vi) $T|_{P\mathcal{H}}$ *is surjective;*

(vii) $I = P + \sum_{j=0}^{\infty} T^j P_0 (T^-)^j$

Proof. Let $P_n = T^n(T^-)^n$, $n \geq 1$. We prove that P_n is the orthogonal projection of $\mathcal{H}$ on $T^n\mathcal{H}$, $n \geq 1$. Indeed,

$$P_1^2 = T(T^-T)T^- = TT^- = P_1$$

$$P_1^* = (T^-)^* T^* = T(T^*T)^{-1}T^* = TT^- = P_1.$$

Hence P_1 is the orthogonal projection of $\mathcal{H}$ on $P_1\mathcal{H} = TT^-\mathcal{H} = T\mathcal{H}$ since T^- is surjective. For $n \geq 2$, T^n is also left invertible.

From the above result it follows that $P_n = T^n(T^n)^- = T^n(T^-)^n$ is the orthogonal projection of $\mathcal{H}$ on $T^n\mathcal{H}$.

It is clear that $P_0 : I - TT^-$ is the orthogonal projection of $\mathcal{H}$ on $\mathcal{H} \ominus T\mathcal{H}$.

We prove that $P_n h \to Ph$, for all $h \in \mathcal{H}$ and $T^m W \perp T^n W, \quad m \neq n$.
It is clear that $P_n - P_{n+1}$ is the orthogonal projection of $\mathcal{H}$ on $T^n\mathcal{H} \bigcap (T^{n+1}\mathcal{H})^\perp$.

It follows

$$(P_n - P_{n+1})\mathcal{H} \perp (P_m - P_{m+1})\mathcal{H}, m \neq n.$$

Hence

$$\begin{aligned}\sum_{n=0}^{m} \|P_n h - P_{n+1}h\|^2 &= \|\sum_{n=0}^{m}(P_n h - P_{n+1}h)\|^2 \\ &= \|h - P_{m+1}h\|^2 \leq 4\|h\|^2\end{aligned}$$

It follows that $\sum_{n=0}^{\infty} \|P_n h - P_{n+1}h\|^2$ converges, i.e. for every $\varepsilon > 0$, there exists an $N(\varepsilon)$ such that for $n \geq N(\varepsilon)$, we have:

$$\begin{aligned}&\|P_n h - P_{n+1}h\|^2 + \|P_{n+1}h - P_{n+2}h\|^2 + \ldots + \|P_{n+p-1}h - P_{n+p}h\|^2 < \varepsilon \\ &\iff \|P_n h - P_{n+p}h\|^2 < \varepsilon, \text{ for all } p \in \mathbb{N}, p \geq 1.\end{aligned}$$

So $(P_n h)$ converges to an element in $\mathcal{H}$.

We denote $Ph = \lim_{n\to\infty} P_n h$. We prove that P is the orthogonal projection of $\mathcal{H}$ on $\bigcap_{n=1}^{\infty} T^n\mathcal{H}$.

Indeed, we consider $h \in \bigcap_{n=1}^{\infty} T^n\mathcal{H}$. Then $h \in T^n\mathcal{H}$, for all $n \geq 1$ and $P_n h = h$, for all $n \geq 1$. Hence $Ph = h$.

On the other hand, if we take $h \perp \bigcap_{n=1}^{\infty} T^n\mathcal{H}$. Notice that, by the definition of P, it follows that $Ph \in \bigcap_{n=1}^{\infty} T^n\mathcal{H}$. Then

$$P(Ph) = \lim_{n\to\infty} P_n(Ph) = \lim_{n\to\infty} Ph = Ph$$

and hence

$$\|Ph\|^2 = \langle Ph, Ph\rangle = \langle P^2h, h\rangle = \langle Ph, h\rangle = 0.$$

Thus $Ph = 0$.

We have

$$\begin{aligned} P_n - P_{n+1} &= T^n(T^-)^n - T^{n+1}(T^-)^{n+1} \\ &= T^n(I - TT^-)(T^-)^n \\ &= T^nP_0(T^-)^n \end{aligned}$$

and

$$(I - P_1) + (P_1 - P_2) + \ldots + (P_n - P_{n+1}) = I - P_{n+1}.$$

Hence $\sum_{n=0}^{\infty} T^nP_0(T^-)^n$ converges to $I - P = Q$.

For proving (iv), we observe that

$$h \in Q\mathcal{H} \Longleftrightarrow Ph = 0 \Longleftrightarrow \lim_{n\to\infty} \|T^n(T^-)^nh\| = 0.$$

The last equality is equivalent with

$$\lim_{n\to\infty} \langle T^n(T^{*n}T^n)^{-1}T^{*n}h, T^n(T^{*n}T^n)^{-1}T^{*n}h\rangle = 0$$

i.e. $\lim_{n\to\infty} \|(T^{*n}T^n)^{-1/2}T^{*n}h\| = 0$.

For proving that $\mathcal{H}_\infty$ reduces T we note that

$$\begin{aligned} P_{n+1}T &= T^{n+1}(T^-)^{n+1}T \\ &= T^{n+1}(T^-)^n(T^-T) = T^{n+1}(T^-)^n \\ TP_n &= TT^n(T^-)^n = T^{n+1}(T^-)^n \end{aligned}$$

Hence $P_{n+1}T = TP_n \Rightarrow PT = TP$ hence $\mathcal{H}_\infty$ reduces T.

Next we prove now that $T|_{\mathcal{H}_\infty}$ is surjective.

Let $h_0 \in \mathcal{H}_\infty$. It follows $h_0 \in T^n\mathcal{H},$ for all $n \geq 1$.

For any $n \geq 1$ there exists $h_n \in \mathcal{H}$ so that $h_0 = T^nh_n$. Then

$$h_0 = Th'_n, h'_n \in T^{n-1}\mathcal{H}, n \geq 1 \Rightarrow T^-h_0 = h'_n, n \geq 1 \Rightarrow h'_n = h'_1, n \geq 1.$$

Hence

$$h_0 = Th'_1, h'_1 \in T^{n-1}\mathcal{H}, n \geq 1 \Rightarrow h'_1 \in \bigcap_{n\geq 1} T^{n-1}\mathcal{H} = \mathcal{H}_\infty.$$

□

Theorem 2.10. *Let $T \in \mathcal{D}$. Then $W := \mathcal{H} \ominus T\mathcal{H}$ is a wandering subspace of $\mathcal{H}$ and*

$$\mathcal{H} = \mathcal{H}_\infty \oplus \mathcal{H}_s,$$

where

$$\mathcal{H}_\infty = \bigcap_{n=1}^{\infty} T^n\mathcal{H}, \quad \mathcal{H}_s = \bigoplus_{n=0}^{\infty} T^n W.$$

$\mathcal{H}_\infty$ and $\mathcal{H}_s$ are reducing spaces of T and $T|_{\mathcal{H}_\infty}$ is bijective. The decomposition is unique.

Proof. The fact that the decomposition exists follows from Theorem 2.9. We prove that the decomposition is unique. Let $\mathcal{H} = \mathcal{H}'_\infty \bigoplus \mathcal{H}'_s$ a decomposition such that

$$T\mathcal{H}'_\infty = \mathcal{H}'_\infty$$

$$\mathcal{H}'_s = \bigoplus_{n=0}^{\infty} T^n W',$$

where W' is a wandering subspace of T. We prove that $\mathcal{H}'_\infty = \mathcal{H}_\infty$ and $\mathcal{H}'_s = \mathcal{H}_s$. Indeed,

$$\begin{aligned} W &= \mathcal{H} \ominus T\mathcal{H} = (\mathcal{H}'_\infty \bigoplus \mathcal{H}'_s) \ominus (T\mathcal{H}'_\infty \bigoplus T\mathcal{H}'_s) \\ &= (\mathcal{H}'_\infty \bigoplus \mathcal{H}'_s) \ominus (\mathcal{H}'_\infty \bigoplus T\mathcal{H}'_s) \\ &= \mathcal{H}'_s \ominus T\mathcal{H}'_s = W' \end{aligned}$$

We use the following fact:
If $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ is such that $\mathcal{H}_1$, $\mathcal{H}_2$ are reducing subspaces of T, then

$$T\mathcal{H} = T\mathcal{H}_1 \oplus T\mathcal{H}_2.$$

This is clear because

$$h = h_1 + h_2, \quad h_1 \perp h_2, \quad h_1 \in \mathcal{H}_1, h_2 \in \mathcal{H}_2$$

$Th = Th_1 + Th_2$ and $\langle Th_1, Th_2\rangle = \langle T^*Th_1, h_2\rangle = 0$. □

The following result is a Wold-type decomposition for a pair of double commuting operators in $\mathcal{D}$.

Theorem 2.11. *Let be $T_1, T_2 \in \mathcal{D}$ double commuting. Then $\mathcal{H}$ has the following orthogonal decomposition*

$$\mathcal{H} = \mathcal{H}_{\infty\infty} \bigoplus \mathcal{H}_{\infty s} \bigoplus \mathcal{H}_{s\infty} \bigoplus \mathcal{H}_{ss},$$

where $\mathcal{H}_{\infty\infty}$, $\mathcal{H}_{\infty s}$, $\mathcal{H}_{s\infty}$, $\mathcal{H}_{ss}$ *are reducing spaces of* $T_i (i = 1, 2)$ *and*

$$\begin{aligned}
\mathcal{H}_{\infty\infty} &= \bigcap_{m,n=1}^{\infty} T_1^m T_2^n \mathcal{H} \\
\mathcal{H}_{\infty s} &= \left(\bigcap_{m=1}^{\infty} T_1^m \mathcal{H} \right) \bigcap \left(\bigoplus_{n=0}^{\infty} T_2^n W_2 \right) \\
\mathcal{H}_{s\infty} &= \left(\bigoplus_{m=0}^{\infty} T_1^m W_1 \right) \bigcap \left(\bigcap_{n=1}^{\infty} T_2^n \mathcal{H} \right) \\
\mathcal{H}_{ss} &= \left(\bigoplus_{m=0}^{\infty} T_1^m W_1 \right) \bigcap \left(\bigoplus_{n=0}^{\infty} T_2^n W_2 \right).
\end{aligned}$$

Proof. We denote by Q_i the orthogonal projection on

$$\bigcap_{n=1}^{\infty} T_i^n \mathcal{H}, \ (i = 1, 2)$$

From hypothesis, Q_1, Q_2 are commuting. The decomposition given in Theorem 2.11 follows from the identity

$$I = Q_1 Q_2 + Q_1 (I - Q_2) + (I - Q_1) Q_2 + (I - Q_1)(I - Q_2).$$

□

Comments. The results of this paper appeared on Arxiv, in 2017, under my maiden name, Laura Găvruţa: https://arxiv.org/abs/1704.04200.

After the paper was posted on Arxiv, we saw the paper [4] and the book [1], in connection with the subject of our paper.

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SECTION METHOD AND FRÉCHET POLYNOMIALS

Dan M. DĂIANU

Abstract

Using the section method we characterize the solutions $f : U \to Y$ of the following four equations

$$\sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} f\left(\sqrt[m]{u^m + iv^m}\right) = (n!)\, f(v),$$

$$f(u) + \sum_{i=1}^{n+1} (-1)^{i} \binom{n+1}{i} f\left(\sqrt[m]{u^m + iv^m}\right) = 0,$$

$$\sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} f\left(\arcsin\left|\sin u \sin^i v\right|\right) = (n!)\, f(v) \text{ and}$$

$$f(u) + \sum_{i=1}^{n+1} (-1)^{i} \binom{n+1}{i} f\left(\arcsin\left|\sin u \sin^i v\right|\right) = 0,$$

where $m \geq 2$ and n are positive integers, $U \subseteq \mathbb{R}$ is a maximally relevant real domain and $(Y, +)$ is an $(n!)$ -divisible Abelian group. [1]

1 Introduction

Educated at the Cluj school of functional equations, approximations and convexity founded by Academician Tiberiu Popoviciu, Professor Borislav Crstici had among his main concerns the functional characterization of polynomials and their generalizations. Without pretending exhaustiveness, we mention here Professor's thesis dedicated to the functional equations that define polynomials [7] and the additions made in [18], [19] and [8]. A brief presentation of Professor's personality

[1] Mathematical Subject Classification (2020): 39A70, 39B52, 47B39
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is given in [6]. In this context we mention the works [1], [2], [3], [4], [5], [9], [10], [11], [12], [13], [20] and [21], published only in the last decade and which contain some generalizations and analyses of different types of polynomials.

Let m, n be positive integers, $m \geq 2$ and $(Y, +)$ be an $(n!)$-divisible Abelian group - i.e. the group homomorphism $Y \to Y$, $y \mapsto (n!)\, y$ is an isomorphism. This paper is dedicated to characterize the solutions $f : \mathbb{R} \to Y$ of the equations

$$\sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} f\left(\sqrt[m]{u^m + iv^m}\right) = (n!)\, f(v) \text{ for all } u, v \in \mathbb{R}, \tag{1}$$

$$f(u) + \sum_{i=1}^{n+1} (-1)^{i} \binom{n+1}{i} f\left(\sqrt[m]{u^m + iv^m}\right) = 0 \text{ for all } u, v \in \mathbb{R}, \tag{2}$$

and the solutions $f : U := \mathbb{R} \setminus \{k\pi|\ k \in \mathbb{Z}\} \to Y$ of the equations

$$\sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} f\left(\arcsin\left|\sin u \sin^{i} v\right|\right) = (n!)\, f(v) \text{ for all } u, v \in U, \tag{3}$$

$$f(u) + \sum_{i=1}^{n+1} (-1)^{i} \binom{n+1}{i} f\left(\arcsin\left|\sin u \sin^{i} v\right|\right) = 0 \text{ for all } u, v \in U. \tag{4}$$

The tool use for these characterizations is the section method [14], [15].

2 Framework

Everywhere in what follows $(X, +)$ is a commutative semigroup, n is a positive integer and $(Y, +)$ is an $(n!)$-divisible Abelian group. We denote by $\mathcal{S}_i$ the set of the solutions of equation (i); for instance $\mathcal{S}_1$ is the set of all functions $f : \mathbb{R} \to Y$ that satisfy equation (1). Let j be a nonnegative integer; we will use the operator

$$\Delta_y^j : Y^X \to Y^X,\ \Delta_y^j \rho(x) := \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} \rho(x + iy)$$

for $y \in X$; $\mathcal{M}_j(X, Y)$ denotes the *j-monomials*, i.e. the solutions $\rho : X \to Y$ of the equation

$$\Delta_y^j \rho(x) = (j!)\, \rho(y) \text{ for all } x, y \in X$$

and $\mathcal{P}_n(X, Y)$ denotes the (*Fréchet*) *n-polynomials*, i.e. the solutions $\rho : X \to Y$ of the equation

$$\Delta_y^{n+1} \rho(x) = 0 \text{ for all } x, y \in X.$$

The first characterization of continuous real n-polynomials by this equation was realized by Fréchet in [17]. A detailed analysis of Fréchet polynomials in the present framework was given by Djoković in [16]; from this paper we will use only the following result.

Lemma 2.1. *Let* $\rho : X \to Y$. *Then* $\rho \in \mathcal{P}_n(X,Y)$ *if and only if there exists* $\rho_j \in \mathcal{M}_j(X,Y)$ *for all* $j \in \{0,1,...,n\}$ *such that* $\rho = \sum_{j=0}^{n} \rho_j$.

The section method [14], [15] provides, among other things, a technique for solving - partial or total - some equations whose solutions are composite functions. We will give below only a few rudiments of this method adapted to our goals.

Let $g : U \to X$ be a surjection and $g' : X \to U$ be a section of g (i.e. $g \circ g' = id_X$). Let also the functions

$$G : Y^X \times X^2 \to Y,\; H : Y \times Y \to Y$$

and the equation

$$G\left(f \circ g', (g(u), g(v))\right) = H(f(u), f(v)) \text{ for all } u, v \in U, \tag{5}$$

where the unknown is $f : U \to Y$. The equation

$$G(\rho, (x,y)) = H(\rho(x), \rho(y)) \text{ for all } x, y \in X, \tag{6}$$

where $\rho : X \to Y$ is the unknown, is *the characteristic* of equation (5).

We will use the following results extracted from Th. 2.4.1, Th 2.4.2 and Th. 2.6.6 in [14].

Lemma 2.2. *1.* $\{f \circ g' | f \in \mathcal{S}_5\} \subseteq \mathcal{S}_6$.

2. $\mathcal{S}_5^c := \{\rho \circ g | \rho \in \mathcal{S}_6\} \subseteq \mathcal{S}_5$.

3. If $f \in \mathcal{S}_5$ *and* $u_0 \in U$ *such that the function*

$$f(U) \to Y,\; y \mapsto H(f(u_0), y) \;\; (\text{or } y \mapsto H(y, f(u_0)))$$

is one-to-one, then $f \in \mathcal{S}_5^c$.

The functions in $\mathcal{S}_5^c$ are named *canonical solutions* (of equation (5)). Thus Lemma 2.2.2 gives a partial solution for equation (5) and Lemma 2.2.3 provides sufficient conditions under which a solution of equation (5) is a canonical solution.

In the following we will use the notions and the conventions introduced above.

3 Radical-Fréchet equations

Let $m \geq 2$ be an integer, $U = \mathbb{R}$ and $(X,+)$ be the additive semigroup defined by

$$(X,+) := \begin{cases} (\mathbb{R},+) & \text{if } m \text{ is odd} \\ (\mathbb{R}_+,+) & \text{if } m \text{ is even} \end{cases},$$

where $\mathbb{R}_+ := [0,\infty)$.

First we characterize the solutions of the *radical-monomial* equation (1).

Theorem 3.1. *Let $f:\mathbb{R}\to Y$ be a function. Then f is a solution of equation (1) if and only if there exists an n-monomial $\rho\in\mathcal{M}_n(X,Y)$ such that*

$$f(u)=\rho(u^m)\ \text{for all } u\in\mathbb{R}. \tag{7}$$

Proof. We apply the section method for the surjection

$$g:\mathbb{R}\to X,\ u\mapsto u^m,$$

its section

$$g':X\to\mathbb{R},\ x\mapsto\sqrt[m]{x},$$

and the functions

$$G:Y^X\times X^2\to Y,\ G(\rho,(x,y)):=\Delta_y^n\rho(x),$$

$$H:Y\times Y\to Y,\ \ H(y_1,y_2):=(n!)\,y_2.$$

Then equation (1) becomes equation (5) and its characteristic is equation (6). Therefore the characteristic of equation (1) is exactly the n-monomial equation

$$\Delta_y^n\rho(x)=(n!)\,\rho(y)\ \text{ for all } x,y\in X$$

and all its solutions are in $\mathcal{M}_n(X,Y)$.

1. Let $\rho\in\mathcal{M}_n(X,Y)$ and $f:\mathbb{R}\to Y$ defined by (7). Then $f=\rho\circ g$ and, according to Lemma 2.2.2, f is a solution of equation (1).

2. For proving the converse it suffices to show that $\mathcal{S}_1\subseteq\mathcal{S}_1^c$. Let f be a solution of equation (1). Since $(Y,+)$ is $(n!)$-divisible, the function

$$f(\mathbb{R})\to Y,\ y\mapsto H(0,y)=(n!)\,y$$

is injective. According to Lemma 2.2.3, f is a canonical solution of equation (1), i.e. there exists $\rho\in\mathcal{M}_n(X,Y)$ such that $f(u)=\rho(u^m)$ for all $u\in\mathbb{R}$. □

Now we are in position to characterize the solutions of the *radical-Fréchet* equation (2).

Theorem 3.2. *Let $f:\mathbb{R}\to Y$ be a function. Then f is a solution of equation (2) if and only if there exists $\rho_i\in\mathcal{M}_i(X,Y)$ for all $i\in\{0,1,...,n\}$ such that*

$$f(u)=\rho_0(u^m)+\rho_1(u^m)+\cdots+\rho_n(u^m)\ \text{for all } u\in\mathbb{R}. \tag{8}$$

Proof. As in the proof of the previous theorem, let

$$g:\mathbb{R}\to X,\ g(u):=u^m \text{ and } g':X\to\mathbb{R},\ g'(x):=\sqrt[m]{x}.$$

Let also be the functions

$$G : Y^X \times X^2 \to Y,\, G(\rho, (x, y)) := \sum_{i=1}^{n+1} (-1)^i \binom{n+1}{i} \rho(x + iy) \text{ and}$$

$$H : Y \times Y \to Y, \;\; H(y_1, y_2) := -y_1.$$

Then equation (2) can be written in the form (5) and its characteristic is (6) or, equivalent, the Fréchet polynomial equation

$$\Delta_y^{n+1} \rho(x) = 0 \text{ for all } x, y \in X.$$

We note that the solutions of the last equation are given by $\mathcal{P}_n(X, Y)$ and their characterization is given by Lemma 2.1.

1. Let $\rho_i \in \mathcal{M}_i(X, Y)$ for all $i \in \{0, 1, ..., n\}$ and $f : \mathbb{R} \to Y$ defined by (8). Then $\rho := \sum_{i=0}^{n} \rho_i \in \mathcal{P}_n(X, Y)$ (by Lemma 2.1) and $f = \rho \circ g \in \mathcal{S}_2$ (by Lemma 2.2.2).

2. Let $f \in \mathcal{S}_2$. To show that f can be expressed by (8) with $\rho_i \in \mathcal{M}_i(X, Y)$ - or, equivalent, that $f = \rho \circ g$, where $\rho := \sum_{i=0}^{n} \rho_i \in \mathcal{P}_n(X, Y)$ -, it is sufficient to prove that f is a canonical solution of equation (2). But the function

$$f(\mathbb{R}) \to Y,\, y \mapsto H(y, 0) = -y$$

is injective; according to Lemma 2.2.3, $f \in \mathcal{S}_2^c$, and the theorem is completely proved. □

4 Arcsine-Fréchet equations

Before proceeding to the characterizations of the solutions of the *arcsine-Fréchet* equations (3) and (4), let us note that $U := \mathbb{R} \setminus \{k\pi | k \in \mathbb{Z}\}$ is the maximal domain on which these equations has nontrivial solutions; indeed if there is $k \in \mathbb{Z}$ such that $k\pi$ is in the domain, for $u = k\pi$ in (3) we get $0 = (n!) f(v)$, and, since $(Y, +)$ is $(n!)$-divisible we immediately obtain $f = 0$, i.e. $\mathcal{S}_3 = \{0\}$; analogously, for $v = k\pi$ in (4) we get $f = 0$ and $\mathcal{S}_4 = \{0\}$.

In the following lines, the set $X := (-\infty, 0]$ is endowed with the addition of real numbers, hence $(X, +)$ is an Abelian semigroup.

Theorem 4.1. *Let $f : U := \mathbb{R} \setminus \{k\pi | k \in \mathbb{Z}\} \to Y$. Then $f \in \mathcal{S}_3$ if and only if there exists an n-monomial $\rho \in \mathcal{M}_n(X, Y)$ and*

$$f(u) = \rho(\ln|\sin u|) \;\; \textit{for all } u \in U. \tag{9}$$

Proof. We apply the section method for

$$g : U \to X,\ g(u) := \ln|\sin u| \text{ for all } u \in U,$$

its section

$$g' : X \to U,\ g'(x) := \arcsin e^{x} \text{ for all } x \in X,$$

and the functions G, H defined by

$$G : Y^{X} \times X^{2} \to Y,\ G(\rho,(x,y)) := \Delta_{y}^{n}\rho(x),$$

$$H : Y \times Y \to Y,\ \ H(y_1,y_2) := (n!)\, y_2.$$

We note that equation (3) becomes equation (5) and its characteristic is equation (6) or, the monomial equation

$$\Delta_{y}^{n}\rho(x) = (n!)\,\rho(y) \text{ for all } x,y \in X.$$

1. If $\rho \in \mathcal{M}_n(X,Y)$ and $f : U \to Y$ is defined by (9), then $f = \rho \circ g$ and - from Lemma 2.2.2 - f is a solution of equation (3).
2. Let $f \in \mathcal{S}_3$ and $u_0 \in U$. Since $(Y,+)$ is $(n!)$-divisible, the function

$$f(U) \to Y,\ y \mapsto H(f(u_0), y) = (n!)\, y$$

is an injection and, from Lemma 2.2.3, there exists an n-monomial $\rho \in \mathcal{M}_n(X,Y)$ such that $f = \rho \circ g$; therefore f satisfies relation (9). □

Finally we characterize the solutions of equation (4).

Theorem 4.2. *Let $f : U \to Y$ be a function. Then f is a solution of equation (4) if and only if there exists the monomials $\rho_i \in \mathcal{M}_i(X,Y)$ for $i \in \{0,1,...,n\}$ such that*

$$f(u) = \sum_{i=0}^{n} \rho_i((\ln|\sin u|)) \text{ for all } u \in U. \tag{10}$$

Proof. Let

$$g : U \to X,\ g(u) := \ln|\sin u| \text{ for all } u \in U,$$

$$g' : X \to U,\ g'(x) := \arcsin e^{x} \text{ for all } x \in X,$$

$$G : Y^{X} \times X^{2} \to Y,\ G(\rho,(x,y)) := \sum_{i=1}^{n+1} (-1)^{i} \binom{n+1}{i} \rho(x+iy) \text{ and}$$

$$H : Y \times Y \to Y,\ H(y_1,y_2) := -y_1.$$

We note that equation (4) can be rewritten in the form (5) and, consequently, its characteristic is equation (6) or, equivalent, the Fréchet polynomial equation

$$\Delta_{y}^{n+1}\rho(x) = 0 \text{ for all } x,y \in X.$$

Then, according to Lemma 2.2.2, $\mathcal{S}_4^c \subseteq \mathcal{S}_4$. Moreover, if $f \in \mathcal{S}_4$ and u_0 is an arbitrary number in U, the function

$$f(U) \to Y,\, y \mapsto H(y, f(u_0)) = -y$$

is bijective; from Lemma 2.2.3 we have $f \in \mathcal{S}_4^c$. Hence

$$\mathcal{S}_4 = \{\rho \circ g |\ \rho \in \mathcal{P}_n(X,Y)\},$$

and, by Lemma 2.1,

$$\mathcal{P}_n(X,Y) = \mathcal{M}_0(X,Y) + \mathcal{M}_1(X,Y) + \cdots + \mathcal{M}_n(X,Y).$$

Consequently, if $f : U \to Y$, then $f \in \mathcal{S}_4$ if and only if there exist $\rho_i \in \mathcal{M}_i(X,Y)$ for $i \in \{0,1,...,n\}$ such that $f(u) = \sum_{i=0}^{n} \rho_i((\ln|\sin u|))$ for all $u \in U$, and the theorem is completely proved. □

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CURRENT DISTRIBUTION IN A MASSIVE COATED HOLLOW CONDUCTOR IN THE PRESENCE OF A FILAMENT

Dragan FILIPOVIĆ, Tatijana DLABAĈ

Abstract

In this paper we present separate rigorous analyses of the skin and proximity effects in an inhomogeneous conductor in the presence of a filament. The conductor consists of a massive hollow circular conductor coated with a thin layer of different conductivity. The skin and proximity solutions are assumed in the form of two infinite sums of the proper harmonics. The unknown coefficients in the skin-effect case are found by applying boundary conditions, while a system of two integral equations is used to determine the unknown coefficients in the proximity-effect case, with no boundary conditions involved. By using the found current density we derive formulas for the conductor power loss.[1]

1 Introduction

There are very few cases where a solution for the current distribution of time-varying currents can be obtained in a closed form. These cases include some conductor configurations in the presence of a filament - massive circular conductor [1-3], thin tubular conductor [4,5], hollow massive conductor [2,6] thin two-layer tubular conductor [7] and inhomogeneous massive solid conductor [8]. The method of integral equations proved to be very powerful in these cases, requiring no boundary conditions when the conductor is homogeneous.

In this paper we investigate the skin and proximity effects in the case of a massive solid hollow circular conductor covered by a thin layer, in the presence of a filament. Although the conductor is inhomogeneous, no boundary conditions are required when analyzing the proximity part of the problem.

[1]Keywords and phrases: skin effect, proximity effect, inhomogeneous conductor, filament, current distribution, integral equation.

2 Formulation of the problem and general form of the solution

Geometry of the problem is shown in Fig. 1. An inhomogeneous conductor consisting of a massive hollow conductor of radii a and b and conductivity σ_1, coated with a thin layer of thickness $d(d << a)$, and conductivity σ_2, and a filament carry equal and opposite sinusoidal currents of rms I and frequency f. Distance between the conductor axis and the filament is D. The object is to find current distribution in the conductor.

The proper radial harmonics in cylindrical coordinates are modified Bessel functions $I_{\mathrm{n}}(kr)$ and $K_{\mathrm{n}}(kr)$ $\left(k^2 = j\omega\mu_0\sigma\right)$, and the proper angular harmonics are trigonometrical functions $\cos n\theta$ and $\sin n\theta$. Due to symmetry, the sine function must be excluded, since the current density should be an even function of θ. Hence, the general form of solution for current density in region 1 is

$$J_1(r,\theta) = \sum_{n=0}^{\infty} \left[A_n I_n\left(k_1 r\right) + B_n K_n\left(k_1 r\right)\right] \cos n\theta \tag{1}$$

In region 2, due to its small thickness, the radial dependence may be neglected, so that general form of solution in this region is

$$J_2(\theta) = \sum_{n=0}^{\infty} C_n \cos n\theta \tag{2}$$

In (1) - (2), A_n, B_n and C_n are unknown coefficients that should be determined.

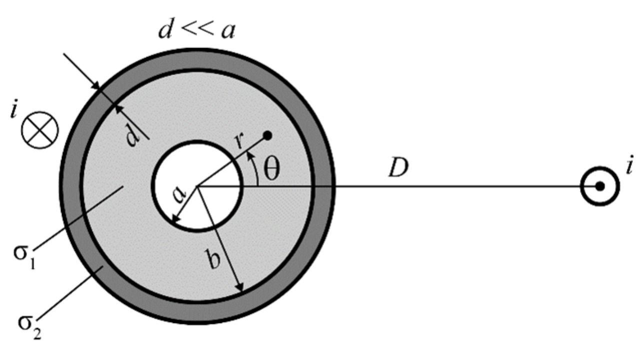

Figure 1: Inhomogeneous hollow conductor and fiament with equal and opposite currents

It is convenient to treat separately the skin and proximity effects in this problem. The skin-effect solutions (the filament is absent) is represented by the first terms ($n = 0$) in (1) – (2), while the remaining infinite sums ($n \geq 1$) account for proximity effect.

2.1 Skin-effect solution

As mentioned above, this solution is given by

$$J_1^S(r) = A_0 I_0\left(k_1 r\right) + B_0 K_0\left(k_1 r\right) \tag{3}$$

$$J_2^S(r) = C_0 = \text{ const.} \tag{4}$$

The unknown coefficients A_0, B_0 and C_0 should be found from the boundary conditions and the known current in the conductor.

Equality of the tangential components of the electrical fields at the interface $r = b$ requires that

$$\frac{1}{\sigma_1}\left(A_0 I_0\left(k_1 b\right) + B_0 K_0\left(k_1 b\right)\right) = \frac{C_0}{\sigma_2} \tag{5}$$

The (tangential) magnetic field at $r = a$ should be zero, hence

$$A_0 I_1\left(k_1 a\right) - B_0 K_1\left(k_1 a\right) = 0 \tag{6}$$

The total current through the conductor is

$$I = \int_{S_1} J_1(r,\theta) r dr d\theta + bd \int_0^{2\pi} J_2(\theta) d\theta$$

where only the first terms ($n = 0$) from (1) - (2) should be taken, since $\int\limits_0^{2\pi} \cos n\theta d\theta = 0, n \geq 1$. Therefore,

$$I = 2\pi \int_a^b \left[A_0 I_0\left(k_1 r\right) + B_0 K_0\left(k_1 r\right) r dr\right] + 2\pi b d C_0.$$

After integration we get

$$\frac{I k_1}{2\pi} = M_0 A_0 + N_0 B_0 + k_1 b d C_0 \tag{7}$$

where

$$M_0 = b I_1\left(k_1 b\right) - a I_1\left(k_1 a\right)$$

$$N_0 = a K_1\left(k_1 a\right) - b K_1\left(k_1 b\right)$$

From (5) - (7) we can find the unknown coefficients

$$A_0 = \frac{Ik_1^2}{2\pi b} \frac{K_1(k_1 a)}{k_1 P_0 + k_2^2 d Q_0} \tag{8}$$

$$B_0 = \frac{Ik_1^2}{2\pi b} \frac{I_1(k_1 a)}{k_1 P_0 + k_2^2 d Q_0} \tag{9}$$

$$C_0 = \frac{Ik_2^2}{2\pi b} \frac{Q_0}{k_1 P_0 + k_2^2 d Q_0} \tag{10}$$

where

$$P_0 = I_1(k_1 b) K_1(k_1 a) - I_1(k_1 a) K_1(k_1 b)$$
$$Q_0 = I_0(k_1 b) K_1(k_1 a) + I_1(k_1 a) K_0(k_1 b)$$

Thus, the skin - effect solution, given by (3) - (4) is completed by (8) - (10).

2.2 Proximity-effect solution

To find the unknown coefficients A_n, B_n and $C_n (n \geq 1)$ in the proximity effect solutions

$$J_1^p(r,\theta) = \sum_{n=1}^{\infty} [A_n I_n(k_1 r) + B_n K_n(k_1 r)] \cos n\theta \tag{11}$$

$$J_2^p(\theta) = \sum_{n=1}^{\infty} C_n \cos n\theta \tag{12}$$

we use the method of integral equations. For the case of two conductors in the presence of a filament these equations have the following form in cylindrical coordinates [9].

$$J_1(r,\theta) = \frac{k_1^2}{4\pi} \left[\sum_{i=1}^{2} \int_{S_i} J_i(r',\theta') \ln f(r,r',\theta,\theta') \, dSi - \right. \tag{13}$$
$$\left. -I \ln f(r,r_0,\theta,\theta_0) \right] + k_1, \quad (r,\theta) \in S_1$$

$$J_2(r,\theta) = \frac{k_2^2}{4\pi} \left[\sum_{i=1}^{2} \int_{S_i} J_i(r',\theta') \ln f(r,r',\theta,\theta') \, dSi - \right. \tag{14}$$
$$\left. -I \ln f(r,r_0,\theta,\theta_0) \right] + K_2, (r,\theta) \in S_2$$

where (r_0, θ_0) specifies the position of the filament, and

$$f\left(r, r', \theta, \theta'\right) = \ln \frac{r^2 + r'^2 - 2rr' \cos\left(\theta - \theta'\right)}{D^2}.$$

In our specific case two conductors are actually the two regions of the inhomogeneous conductor so that $J_1(r,\theta) = J_1^p(r,\theta)$, $J_2(r,\theta) = J_2^p(\theta), dS_1 = r'dr'd\theta', r_0 = \mathrm{D}, \theta_0 = 0$. Since d is very small we may take $dS_2 \approx bdd\theta', r \approx b$ in (14) and $r' \approx b$ in the integral over S_2 in (13). By replacing $J_1(r,\theta)$ and $J_2(r,\theta)$ in (13) - (14) with $J_1^p(r,\theta)$ and $J_2^p(\theta)$ from (11) - (12) we obtain

$$\begin{aligned} &\sum_{n=1}^{\infty} \left[A_n I_n\left(k_1 r\right) + B_n K_n\left(k_1 r\right)\right] \cos n\theta = \\ &= \frac{k_1^2}{4\pi} \left[\sum_{n=1}^{\infty} A_n R_n(r,\theta) + \sum_{n=1}^{\infty} B_n S_n(r,\theta) + \right. \\ &\left. + bd \sum_{n=1}^{\infty} C_n T_n(r,\theta) - I \ln \frac{r^2 + D^2 - 2rD\cos\theta}{D^2}\right] + K_1 \end{aligned} \tag{15}$$

$$\begin{aligned} &\sum_{n=1}^{\infty} C_n \cos n\theta = \\ &= \frac{k_1^2}{4\pi} \left[\sum_{n=1}^{\infty} A_n U_n(r,\theta) + \sum_{n=1}^{\infty} B_n V_n(r,\theta) + \right. \\ &\left. + bd \sum_{n=1}^{\infty} C_n W_n(\theta) - I \ln \frac{b^2 + D^2 - 2bD\cos\theta}{D^2}\right] + K_2 \end{aligned} \tag{16}$$

where R_n, S_n, T_n, U_n, V_n and W_n are some integrals given by

$$R_n(r,\theta) = \int_0^{2\pi} \cos n\theta' d\theta' \int_a^b r' I_n\left(k_1 r'\right) \ln \frac{r^2 + r'^2 - 2rr'\cos\left(\theta - \theta'\right)}{D^2} dr' \tag{17}$$

$$S_n(r,\theta) = \int_0^{2\pi} \cos n\theta' d\theta' \int_a^b r' K_n\left(k_1 r'\right) \ln \frac{r^2 + r'^2 - 2rr'\cos\left(\theta - \theta'\right)}{D^2} dr' \tag{18}$$

$$T_n(r,\theta) = \int_0^{2\pi} \cos n\theta' d\theta' \ln \frac{r^2 + b^2 - 2rb\cos\left(\theta - \theta'\right)}{D^2} d\theta \tag{19}$$

$$U_n(r,\theta) = \int_0^{2\pi} \cos n\theta' d\theta' \int_a^b r' I_n\left(k_1 r'\right) \ln \frac{b^2 + r'^2 - 2br'\cos\left(\theta - \theta'\right)}{D^2} dr' \cong R_n(b,\theta) \tag{20}$$

$$V_n(r,\theta) = \int_0^{2\pi} \cos n\theta' d\theta' \int_a^b r' K_n \left(k_1 r'\right) \ln \frac{b^2 + r'^2 - 2br' \cos\left(\theta - \theta'\right)}{D^2} dr' \cong S_n(b,\theta) \tag{21}$$

$$W_n(r,\theta) = \int_0^{2\pi} \cos n\theta' \ln \frac{4b^2 \sin^2 \frac{(\theta-\theta')}{2}}{D^2} d\theta \equiv T_n(b,\theta) \tag{22}$$

Integrals (17)-(19) have been evaluated earlier. This is done in [6] for R_n and S_n and in [8] for T_n. The remaining three integrals U_n, V_n and W_n follow then from R_n, S_n and T_n respectively by putting $r = b$. All these results (we will not write them down) should be substituted into (15)-(16).

If we now equate the coefficients with $r^n \cos n\theta$ and the coefficients with $r^{-n} \cos n\theta$ on both sides of (15)-(16), the following two equations follow

$$A_n I_{n-1}\left(k_1 b\right) - B_n K_{n-1}\left(k_1 b\right) + C_n k_1 d = \frac{I k_1}{\pi b}\left(\frac{b}{D}\right)^n \tag{23}$$

$$A_n I_{n+1}\left(k_1 a\right) - B_n K_{n+1}\left(k_1 a\right) = 0 \tag{24}$$

The third equation that is necessary for determining the three unknown coefficients is obtained by equating the coefficients with $\cos n\theta$ on both sides of (16). It is:

$$A_n M_n + B_n N_n + C_n k_1 d \left(1 + \frac{n}{j\lambda_2}\right) = \frac{I k_1}{\pi b}\left(\frac{b}{D}\right)^n \tag{25}$$

where

$$j\lambda_2 = \frac{k_2^2 b d}{2}$$

From (23) - (25) we can find the unknown coefficients

$$A_n = \frac{I k_1^2}{\pi b}\left(\frac{b}{D}\right)^n \frac{K_{n+1}\left(k_1 a\right)}{k_1 P_n + k_2^2 d Q_n} \tag{26}$$

$$B_n = \frac{I k_1^2}{\pi b}\left(\frac{b}{D}\right)^n \frac{I_{n+1}\left(k_1 a\right)}{k_1 P_n + k_2^2 d Q_n} \tag{27}$$

$$C_n = \frac{I k_2^2}{\pi b}\left(\frac{b}{D}\right)^n \frac{Q_n}{k_1 P_n + k_2^2 d Q_n} \tag{28}$$

where

$$P_n = I_{n-1}\left(k_1 b\right) K_{n+1}\left(k_1 a\right) - I_{n+1}\left(k_1 a\right) K_{n-1}\left(k_1 b\right) \tag{29}$$

$$Q_n = I_n\left(k_1 b\right) K_{n+1}\left(k_1 a\right) - I_{n+1}\left(k_1 a\right) K_n\left(k_1 b\right) \tag{30}$$

It should be noted that several formulas from the Bessel function theory have to be used in deriving coefficients (26)(28).

A remarkable feature of the integral equation method that we used to determine the unknown coefficients in the proximity effect solution is that no boundary conditions are used, although the conductor is inhomogeneous. Furthermore, it may be shown that (26)-(28) ensure that the appropriate boundary conditions (equality of the tangential components of the electric and magnetic fields at the interface $r = b$) are met.

2.3 Solution for the total current densities

This solution is obtained by summing the skin and proximity-effect solutions

$$J_1(r,\theta) = A_0 I_0\left(k_1 r\right) + B_0 K_0\left(k_1 r\right) + \\ + \sum_{n=1}^{\infty}\left[A_n I_n\left(k_1 r\right) + B_n K_n\left(k_1 r\right)\right]\cos n\theta \tag{31}$$

$$J_2(\theta) = C_0 + \sum_{n=1}^{\infty} C_n \cos n\theta \tag{32}$$

with $A_n, B_n, C_n, n \geq 0$ given by (8) - (10), (26) - (28).

From the obtained results some special cases considered earlies immediately follow: solid hollow conductor and filament [6] (we let $\sigma_2 = 0\,(k_2 = 0)$ in the expressions for $A_n, B_n, C_n, n \geq 0$) and thin two-layer tubular conductor and filament [7] (we let $a = b - d, d << a$).

3 Power loss in the conductor

The total power loss in the conductor is obtained by summing power losses in the conductor core ($a <$ r $< b$) and in the thin coat layer. These losses are calculated by Joule's low.

The power loss per unit lenhght in the conductor core is

$$P'_{J_1} = \frac{1}{\sigma_1} \int_{S_1} \left|J_1\right|^2 r dr d\theta \tag{33}$$

By using (31) we can write

$$|J_1|^2 = J_1 \cdot J_1^* = \sum_{n=0}^{\infty} D_n(r)\cos n\theta \cdot \sum_{n=0}^{\infty} D_n^*(r)\cos n\theta = \tag{34}$$

$$= \sum_{n=0}^{\infty} |D_n(r)|^2 \cos^2 n\theta + \sum_{\substack{m,n=0 \\ (m\neq n)}}^{\infty} D_m(r) D_n^*(r)\cos m\theta \cos n\theta$$

where $D_n(r) = A_n I_n(k_1 r) + B_n K_n(k_1 r)$.

The last summation in (34) does not contribute to the integral in (33), due to ortogonality of the cosine function, hence

$$P'_{J_1} = \frac{1}{\sigma_1}\int_0^{2\pi} \cos^2 n\theta d\theta \int_a^b r\,|D_n(r)|^2\,dr =$$

$$= \frac{2\pi}{\sigma_1}\left(\int_a^b r\,|D_0(r)|^2\,dr + \frac{1}{2}\sum_{n=1}^{\infty}\int_a^b r\,|D_n(r)|^2\,dr\right). \tag{35}$$

The integrals in (35) can be evaluated in a closed form [2], in terms of the Kelvin functions, or alternatively by numerical techniques.

The power loss per unit length in the thin layer is

$$P'_{J_1} = \frac{bd}{\sigma_1}\int_0^{2\pi} |J_2|^2\,d\theta$$

where

$$|J_2|^2 = J_2 \cdot J_2^* = \sum_{n=0}^{\infty} C_n \cos n\theta \cdot \sum_{n=0}^{\infty} C_n^* \cos n\theta =$$

$$= \sum_{n=0}^{\infty} |C_n|^2 \cos^2 n\theta + \sum_{\substack{m,n=0 \\ (m\neq n)}}^{\infty} C_n C_n^* \cos m\theta \cos n\theta \tag{36}$$

As explaned above, we may ignore the last summation in (36), hence

$$P'_{J_2} = \frac{2\pi bd}{\sigma_1}\left(|C_0|^2 + \frac{1}{2}\sum_{n=1}^{\infty}|C_n|^2\right)$$

4 Conclusion

A rigorous analysis of the current distribution in a massive hollow cylindrical conductor with a thin layer in the presence of a filament is presented in this paper. Closed form solutions are found in the two regions in the form of infinite sums of the proper harmonics, and the unknown coefficients are found from boundary conditions in the skin-effect case, and from two integral equations in the proximity-effect case. It is remarkable that no boundary conditions are required in the latter case. Formulas for the power loss in the conductor are also included.

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ON UNIFORM DICHOTOMY WITH DIFFERENTIABLE GROWTH RATES IN BANACH SPACES

Ariana GĂINĂ

Abstract

In this paper is considered the concept of uniform dichotomy with differentiable growth rates for skew-evolution cocycles in Banach spaces which includes as particular case, the well-known concept of uniform exponential dichotomy. Necessary and sufficient conditions are presented using invariant families of projectors. [1]

1 Introduction

The notion of dichotomy represents one of the most significant asymptotic behavior studied for evolution equations. A great number of papers that describe exponential dichotomy as well as dichotomy with growth rates regarding uniform or nonuniform case are published by Dragičević, Sasu and Sasu (2022) [6], Găină, Megan and Popa (2021) [7], Găină (2022) [8], Lupa and Megan (2014) [9], Megan, Sasu and Sasu (2003) [11] , (2004) [12], Megan and Stoica (2009) [13], (2010) [14], Rămneanţu, Ceauşu and Megan (2012) [18], Sasu and Sasu (2019) [19], Sasu (2010) [20].

The uniform dichotomy with differentiable growth rates or uniform strong h-dichotomy considered in this paper is a particular case of the above concepts. This idea of introducing a growth rate is given by Pinto [17]. Also, recently, we remark the results obtained for the notions of growth rates and differentiable growth rates by Bento, Lupa, Megan and Silva (2017) [1], Boruga, Megan and Toth (2021) [2],

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(2022) [3], Găină (2022) [8], Megan, Găină and Boruga (Toma) (2023) [10], Mihiţ, Borlea and Megan (2017) [15], T. Yue (2022) [21] and the references therein.

In this paper, the starting point is due to Datko (1972) [5], who gave an integral characterization of uniform exponential stabilty for the evolution operators. Extending the above work, it was obtained integral characterizations for the concept of uniform dichotomy with differentiable growth rates for skew-evolution cocycles in Banach spaces using invariant families of projectors. Also, the particular case of uniform exponential dichotomy is presented.

2 Preliminaries

We consider X a metric space, V a Banach space and also $\mathcal{B}(V)$ the Banach algebra of all bounded linear operators acting on V.

The following sets will be used throughout the paper:

$$\Delta = \{(t,s) \in \mathbb{R}_+^2 : t \geq s\}$$

$$T = \{(t,s,t_0) \in \mathbb{R}_+^3 : t \geq s \geq t_0\}.$$

Definition 2.1. A mapping $\varphi : \Delta \times X \to X$ is called an *evolution semiflow* on X if:

(es_1) $\varphi(s,s,x) = x$, for all $(s,x) \in \mathbb{R}_+ \times X$;

(es_2) $\varphi(t,s,\varphi(s,t_0,x_0)) = \varphi(t,t_0,x_0)$, for all $(t,s,t_0,x_0) \in T \times X$.

Definition 2.2. A mapping $\Phi : \Delta \times X \to \mathcal{B}(V)$ is called a *skew-evolution semiflow* over the evolution semiflow φ if:

(ses_1) $\Phi(s,s,x) = I$ (the identity operator on V), for all $(s,x) \in \mathbb{R}_+ \times X$;

(ses_2) $\Phi(t,s,\varphi(s,t_0,x_0))\Phi(s,t_0,x_0) = \Phi(t,t_0,x_0)$, for all $(t,s,t_0,x_0) \in T \times X$.

If $\varphi : \Delta \times X \to X$ is an evolution semiflow and $\Phi : \Delta \times X \to \mathcal{B}(V)$ a skew-evolution semiflow over the evolution semiflow φ, then the pair $C = (\Phi, \varphi)$ is called *a skew-evolution cocycle.*

Example 2.3. Let X be a metric space, V a Banach space, $\varphi : \Delta \times X \to X$ an evolution semiflow on X and $A : X \to \mathcal{B}(V)$ a continuous mapping. If $\Phi(t,s,x)$ is the solution of the Cauchy problem

$$\begin{cases} \dot{v}(t) = A(\varphi(t,s,x))v(t), t > s \\ v(s) = x \ , \end{cases}$$

then $C = (\Phi, \varphi)$ is a skew-evolution cocycle.

Definition 2.4. A mapping $P : \mathbb{R}_+ \times X \to \mathcal{B}(V)$ is called a *family of projectors* if

$$P^2(t,x) = P(t,x), \text{ for all } (t,x) \in \mathbb{R}_+ \times X.$$

If $P : \mathbb{R}_+ \times X \to \mathcal{B}(V)$ is a family of projectors, then $Q : \mathbb{R}_+ \times X \to \mathcal{B}(V)$, $Q(s,x) = I - P(s,x)$ is called *the complementary family of projectors of P.*

Definition 2.5. A family of projectors $P : \mathbb{R}_+ \times X \to \mathcal{B}(V)$ is called *invariant* for the skew-evolution cocycle C if the following relation takes place

$$\Phi(t,s,x)P(s,x) = P(t,\varphi(t,s,x))\Phi(t,s,x),$$

for all $(t,s,x) \in \Delta \times X$.

Remark 2.6. If the family of projectors $P : \mathbb{R}_+ \times X \to \mathcal{B}(V)$ is invariant for the skew-evolution cocycle C, then its complementary family of projectors $Q : \mathbb{R}_+ \times X \to \mathcal{B}(V)$ is also invariant for C.

In [10] is proved the following:

Proposition 2.7. If P is invariant for the skew-evolution cocycle C, then the application $\Phi_P : \Delta \times X \to \mathcal{B}(V)$ defined by

$$\Phi_P(t,s,x) = \Phi(t,s,x)P(s,x)$$

has the following properties:

(i) $\Phi_P(t,s,x) = P(t,\varphi(t,s,x))\Phi_P(t,s,x)$, for all $(t,s,x) \in \Delta \times X$;

(ii) $\Phi_P(t,t,x) = P(t,x)$, for all $(t,x) \in \mathbb{R}_+ \times X$;

(iii) $\Phi_P(t,t_0,x_0) = \Phi_P(t,s,\varphi(s,t_0,x_0))\Phi_P(s,t_0,x_0)$, for all $(t,s,t_0,x_0) \in T \times X$.

Definition 2.8. A nondecreasing function $h : \mathbb{R}_+ \to [1,\infty)$ with $\lim_{t\to\infty} h(t) = \infty$ is called a *growth rate.*

Definition 2.9. The pair (C,P) is *uniformly strongly h-dichotomic* if there are $N \geq 1$ and $\nu > 0$ with:

$(ushd_1)$ $h(t)^\nu ||\Phi_P(t,t_0,x_0)v_0|| \leq N^{\frac{h'(s)}{h(s)}} h(s)^\nu ||\Phi_P(s,t_0,x_0)v_0||$;

$(ushd_2)$ $h(t)^\nu ||\Phi_Q(s,t_0,x_0)v_0|| \leq N^{\frac{h'(t)}{h(t)}} h(s)^\nu ||\Phi_Q(t,t_0,x_0)v_0||$,

for all $(t,s,t_0,x_0,v_0) \in T \times X \times V$.

Remark 2.10. In the definition just mentioned, it can be taken $\nu \in (0,1)$.

Remark 2.11. The pair (C,P) is said to be uniformly strongly h-dichotomic if and only if there exist two constants $N \geq 1$ and $\nu > 0$ with:

$(ushd_1')$ $h(t)^\nu ||\Phi_P(t,s,x)v|| \leq N \frac{h'(s)}{h(s)} h(s)^\nu ||P(s,x)v||$;

$(ushd_2')$ $h(t)^\nu ||Q(s,x)v|| \leq N \frac{h'(t)}{h(t)} h(s)^\nu ||\Phi_Q(t,s,x)v||$,

for all $(t,s,x,v) \in \Delta \times X \times V$.

Definition 2.12. The pair (C,P) has *uniform strong h-growth* if there are $M \geq 1, \omega > 0$ with:

$(ushg_1)$ $h(s)^\omega ||\Phi_P(t,t_0,x_0)v_0|| \leq M \frac{h'(s)}{h(s)} h(t)^\omega ||\Phi_P(s,t_0,x_0)v_0||$;

$(ushg_2)$ $h(s)^\omega ||\Phi_Q(s,t_0,x_0)v_0|| \leq M \frac{h'(t)}{h(t)} h(t)^\omega ||\Phi_Q(t,t_0,x_0)v_0||$,

for all $(t,s,t_0,x_0,v_0) \in T \times X \times V$.

Remark 2.13. The pair (C,P) has uniform strong h-growth if and only if there exist $M \geq 1$ and $\omega > 0$ with:

$(ushg_1')$ $h(s)^\omega ||\Phi_P(t,s,x)v|| \leq M \frac{h'(s)}{h(s)} h(t)^\omega ||P(s,x)v||$;

$(ushg_2')$ $h(s)^\omega ||Q(s,x)v|| \leq M \frac{h'(t)}{h(t)} h(t)^\omega ||\Phi_Q(t,s,x)v||$,

for all $(t,s,x,v) \in \Delta \times X \times V$.

In what follows we denote by

• $\mathcal{H}$ the set of functions $h : \mathbb{R}_+ \to [1,\infty)$ with the property that there exists $H > 1$ such that $h'(t) \leq Hh(t)$, for all $t \geq 0$.

• $\mathcal{H}_1$ the set of functions $h : \mathbb{R}_+ \to [1,\infty)$ with the property that there exists $m > 0$ such that $h'(t) \geq mh(t)$, for all $t \geq 0$.

Remark 2.14. If h is an exponential function, then $h \in \mathcal{H} \cap \mathcal{H}_1$.

Proposition 2.15. If $h \in \mathcal{H}$, then:

(i) $h(t) \leq h(s)e^{H(t-s)}$, for all $(t,s) \in \Delta$;

(ii) $h(t+1) \leq e^H h(t)$, for all $t \geq 0$.

Proof. See [10]. □

Remark 2.16. The concept of uniform strong h-dichotomy implies the concept of uniform strong h-growth. The converse implication is not valid, as we can see in the following result.

Example 2.17. We consider $V = \mathbb{R}^2$, $C = (\Phi, \varphi)$ a skew-evolution cocycle with the evolution semiflow $\varphi : \Delta \times X \to X$ and the skew-evolution semiflow $\Phi : \Delta \times X \to \mathcal{B}(V)$ defined by

$$\Phi(t,s,x)v = \left(\frac{h(t)}{h(s)}v_1, \frac{h(s)}{h(t)}v_2\right)$$

and the family of projectors $P, Q : \mathbb{R}_+ \times X \to \mathcal{B}(V)$ with

$$P(s,x)v = (v_1, 0) \text{ and } Q(s,x)v = (0, v_2),$$

where $v = (v_1, v_2)$ as in *Example 2.3* from [8]. Besides these, we take $h \in \mathcal{H}_1$.

It is immediate that the pair (C, P) has uniform strong h-growth, for $\omega = 2$ and $M = \frac{1}{m}$, where m is given by the definition of $\mathcal{H}_1$.

If we assume that the pair (C, P) is uniformly strongly h-dichotomic, then it results that there exist $N \geq 1$ and $\nu > 0$ such that

$$||\Phi_P(t,s,x)v|| \leq N\left(\frac{h(t)}{h(s)}\right)^{-\nu}\frac{h'(s)}{h(s)}||P(s,x)v||.$$

We obtain

$$h(t)^{\nu+1} \leq NHh(s)^{\nu}h'(s).$$

For $s = 0$ and $t \to \infty$ we obtain contradiction, so (C, P) is not an uniformly strongly h-dichotomic pair.

Remark 2.18. If $h(t) = e^t$ in *Definition 2.9* and *Definition 2.12*, then we obtain the concepts of *uniform exponential dichotomy*, respectively *uniform exponential growth*.

3 Main results

Definition 3.1. The skew-evolution cocycle $C = (\Phi, \varphi)$ is said to be *strongly measurable* if the mapping $t \mapsto ||\Phi(t,s,x)v||$ is measurable on $[s, \infty)$, for all $(s,x,v) \in \mathbb{R}_+ \times X \times V$.

In the following theorems we consider $C = (\Phi, \varphi)$ a strongly measurable skew-evolution cocycle and P an invariant family of projectors.

Theorem 3.2. We consider that (C, P) has uniform strong h-growth with $h \in \mathcal{H} \cap \mathcal{H}_1$. The pair (C, P) is uniformly strongly h-dichotomic if and only if there exist $D > 1$ and $d \in (0, 1)$ with

$$(ushD_1)\quad \int_t^\infty \frac{h'(\tau)}{h(\tau)}h(\tau)^d||\Phi_P(\tau,t_0,x_0)v_0||d\tau \leq Dh(t)^d\frac{h'(t)}{h(t)}||\Phi_P(t,t_0,x_0)v_0||,$$

$(ushD_2)$ $\int\limits_t^\infty \frac{h'(\tau)}{h(\tau)} \frac{h(\tau)^d}{||\Phi_Q(\tau,t_0,x_0)v_0||} d\tau \leq \frac{h'(t)}{h(t)} \frac{Dh(t)^d}{||\Phi_Q(t,t_0,x_0)v_0||}$ with $Q(t_0,x_0)v_0 \neq 0$,

for all $(t,t_0,x_0,v_0) \in \Delta \times X \times V$.

Proof. Necessity. We consider $d \in (0,\nu)$. The relations $(ushd_1) \implies (ushD_1)$ and $(ushd_2) \implies (ushD_2)$ result from *Definition 2.9* :

$$\int\limits_t^\infty \frac{h'(\tau)}{h(\tau)} h(\tau)^d ||\Phi_P(\tau,t_0,x_0)v_0|| d\tau \leq$$

$$\leq N \int\limits_t^\infty \frac{h'(\tau)}{h(\tau)} h(\tau)^d \left(\frac{h(\tau)}{h(t)}\right)^{-\nu} \frac{h'(t)}{h(t)} ||\Phi_P(t,t_0,x_0)v_0|| d\tau \leq$$

$$\leq N h(t)^\nu \frac{h'(t)}{h(t)} ||\Phi_P(t,t_0,x_0)v_0|| \int\limits_t^\infty h'(\tau) h(\tau)^{d-\nu-1} d\tau \leq$$

$$\leq \frac{N}{\nu - d} h(t)^d \frac{h'(t)}{h(t)} ||\Phi_P(t,t_0,x_0)v_0||,$$

respectively

$$\int\limits_t^\infty \frac{h'(\tau)}{h(\tau)} \frac{h(\tau)^d}{||\Phi_Q(\tau,t_0,x_0)v_0||} d\tau \leq N \int\limits_t^\infty \frac{h'(\tau)}{h(\tau)} \left(\frac{h(\tau)}{h(t)}\right)^{-\nu} \frac{h'(\tau)}{h(\tau)} \frac{h(\tau)^d}{||\Phi_Q(t,t_0,x_0)v_0||} d\tau =$$

$$= \frac{NHh(t)^\nu}{||\Phi_Q(t,t_0,x_0)v_0||} \int\limits_t^\infty h'(\tau) h(\tau)^{d-\nu-1} d\tau \leq$$

$$\leq \frac{NH}{m(\nu - d)} \frac{h'(t)}{h(t)} \frac{h(t)^d}{||\Phi_Q(t,t_0,x_0)v_0||}.$$

Sufficiency. In order to obtain the concept of uniform strong h-dichotomy we use two cases, one for the relation $(ushD_1) \implies (ushd_1)$:

Case 1. $t \geq s+1$.

$$||\Phi_P(t,t_0,x_0)v_0|| = \int\limits_{t-1}^t ||\Phi_P(t,t_0,x_0)v_0|| d\tau \leq$$

$$\leq M\int_{t-1}^{t}\left(\frac{h(t)}{h(\tau)}\right)^{\omega}\frac{h'(\tau)}{h(\tau)}||\Phi_P(\tau,t_0,x_0)v_0||d\tau =$$

$$= M\int_{t-1}^{t}\left(\frac{h(t)}{h(s)}\right)^{-d}\left(\frac{h(t)}{h(\tau)}\right)^{d+\omega}\frac{h'(\tau)}{h(\tau)}\left(\frac{h(\tau)}{h(s)}\right)^{d}||\Phi_P(\tau,t_0,x_0)v_0||d\tau \leq$$

$$\leq Me^{H(\omega+d)}\left(\frac{h(t)}{h(s)}\right)^{-d}\int_{s}^{\infty}\frac{h'(\tau)}{h(\tau)}\left(\frac{h(\tau)}{h(s)}\right)^{d}||\Phi_P(\tau,t_0,x_0)v_0||d\tau \leq$$

$$\leq DMe^{H(\omega+d)}\left(\frac{h(t)}{h(s)}\right)^{-d}\frac{h'(s)}{h(s)}||\Phi_P(s,t_0,x_0)v_0||.$$

Case 2. $t \in [s, s+1)$.

$$\begin{aligned}||\Phi_P(t,t_0,x_0)v_0|| &\leq M\left(\frac{h(t)}{h(s)}\right)^{\omega}\frac{h'(s)}{h(s)}||\Phi_P(s,t_0,x_0)v_0|| =\\ &= M\left(\frac{h(t)}{h(s)}\right)^{\omega+d}\left(\frac{h(t)}{h(s)}\right)^{-d}\frac{h'(s)}{h(s)}||\Phi_P(s,t_0,x_0)v_0|| \leq\\ &\leq Me^{H(\omega+d)}\left(\frac{h(t)}{h(s)}\right)^{-d}\frac{h'(s)}{h(s)}||\Phi_P(s,t_0,x_0)v_0||.\end{aligned}$$

and another one for $(ushD_2) \implies (ushd_2)$:
Case 1: $t \geq s+1$ *and* $Q(t_0,x_0)v_0 \neq 0$.

$$\begin{aligned}\frac{h(t)^d}{||\Phi_Q(t,t_0,x_0)v_0||} &= \int_{t-1}^{t}\frac{h(t)^d}{||\Phi_Q(t,t_0,x_0)v_0||}d\tau \leq\\ &\leq M\int_{t-1}^{t}\frac{h'(t)}{h(t)}\left(\frac{h(t)}{h(\tau)}\right)^{\omega}\frac{h(t)^d}{||\Phi_Q(\tau,t_0,x_0)v_0||}d\tau \leq\\ &\leq \frac{M}{m}\frac{h'(t)}{h(t)}\int_{t-1}^{t}\left(\frac{h(t)}{h(\tau)}\right)^{\omega+d}\frac{h'(\tau)}{h(\tau)}\frac{h(\tau)^d}{||\Phi_Q(\tau,t_0,x_0)v_0||}d\tau \leq\\ &\leq \frac{M}{m}e^{H(\omega+d)}\frac{h'(t)}{h(t)}\int_{s}^{\infty}\frac{h'(\tau)}{h(\tau)}\frac{h(\tau)^d}{||\Phi_Q(\tau,t_0,x_0)v_0||}d\tau \leq\\ &\leq \frac{DM}{m}e^{H(\omega+d)}\frac{h'(t)}{h(t)}\frac{h'(s)}{h(s)}\frac{h(s)^d}{||\Phi_Q(s,t_0,x_0)v_0||} \leq\\ &\leq \frac{DMH}{m}e^{H(\omega+d)}\frac{h'(t)}{h(t)}\frac{h(s)^d}{||\Phi_Q(s,t_0,x_0)v_0||}\end{aligned}$$

Case 2. $t \in [s, s+1)$

$$\begin{aligned} ||\Phi_Q(s,t_0,x_0)v_0|| &\leq M\left(\frac{h(t)}{h(s)}\right)^{\omega}\frac{h'(t)}{h(t)}||\Phi_Q(t,t_0,x_0)v_0|| = \\ &= M\left(\frac{h(t)}{h(s)}\right)^{\omega+d}\left(\frac{h(t)}{h(s)}\right)^{-d}\frac{h'(t)}{h(t)}||\Phi_Q(t,t_0,x_0)v_0|| \leq \\ &\leq Me^{H(\omega+d)}\left(\frac{h(t)}{h(s)}\right)^{-d}\frac{h'(t)}{h(t)}||\Phi_Q(t,t_0,x_0)v_0||. \end{aligned}$$

□

Theorem 3.3. Let us consider the pair (C, P) that has uniform strong h-growth, where $h \in \mathcal{H} \cap \mathcal{H}_1$. The pair (C, P) is uniformly strongly h-dichotomic if and only if there exist the constants $D > 1$ and $d \in (0, 1)$ such that

$(ushD_1')$ $\int\limits_{t_0}^{t} \frac{h'(s)}{h(s)}\frac{h(s)^{-d}}{||\Phi_P(s,t_0,x_0)v_0||}ds \leq \frac{h'(t)}{h(t)}\frac{Dh(t)^{-d}}{||\Phi_P(t,t_0,x_0)v_0||}$ with $\Phi_P(t,t_0,x_0)v_0 \neq 0$;

$(ushD_2')$ $\int\limits_{t_0}^{t} \frac{h'(\tau)}{h(\tau)}h(\tau)^{-d}||\Phi_Q(\tau,t_0,x_0)v_0||d\tau \leq D\frac{h'(t)}{h(t)}h(t)^{-d}||\Phi_Q(t,t_0,x_0)v_0||$,
for all $(t, t_0, x_0, v_0) \in \Delta \times X \times V$.

Proof. Necessity. Let d be from $(0, \nu)$.

$$\begin{aligned} \int\limits_{t_0}^{t}\frac{h'(s)}{h(s)}\frac{h(s)^{-d}}{||\Phi_P(s,t_0,x_0)v_0||}ds &\leq N\int\limits_{t_0}^{t}\frac{h'(s)}{h(s)}\left(\frac{h(t)}{h(s)}\right)^{-\nu}\frac{h'(s)}{h(s)}\frac{h(s)^{-d}}{||\Phi_P(t,t_0,x_0)v_0||}ds \leq \\ &\leq \frac{NHh(t)^{-\nu}}{||\Phi_P(t,t_0,x_0)v_0||}\int\limits_{t_0}^{t}h'(s)h(s)^{\nu-d-1}ds \leq \\ &\leq \frac{NH}{m(\nu-d)}\frac{h'(t)}{h(t)}\frac{h(t)^{-d}}{||\Phi_P(t,t_0,x_0)v_0||}. \end{aligned}$$

In a similar way we obtain:

$$\int_{t_0}^{t} \frac{h'(\tau)}{h(\tau)} h(\tau)^{-d} ||\Phi_Q(\tau, t_0, x_0)v_0|| d\tau \leq$$

$$\leq N \int_{t_0}^{t} \frac{h'(\tau)}{h(\tau)} h(\tau)^{-d} \left(\frac{h(t)}{h(\tau)}\right)^{-\nu} \frac{h'(t)}{h(t)} ||\Phi_Q(t, t_0, x_0)v_0|| d\tau =$$

$$= N h(t)^{-\nu} \frac{h'(t)}{h(t)} ||\Phi_Q(t, t_0, x_0)v_0|| \int_{t}^{\infty} h'(\tau) h(\tau)^{\nu-d-1} d\tau \leq$$

$$\leq \frac{N}{\nu - d} h(t)^{-d} \frac{h'(t)}{h(t)} ||\Phi_Q(t, t_0, x_0)v_0||.$$

Sufficiency. By *Definition 2.12* and relations $(ushD_1')$ respectively $(ushD_2')$ we have

Case 1: $t \geq s + 1$ *and* $\Phi_P(t, t_0, x_0)v_0 \neq 0$.

$$\frac{h(s)^{-d}}{||\Phi_P(s, t_0, x_0)v_0||} = \int_{s}^{s+1} \frac{h(s)^{-d}}{||\Phi_P(s, t_0, x_0)v_0||} d\tau \leq$$

$$\leq M \int_{s}^{s+1} \frac{h'(s)}{h(s)} \left(\frac{h(\tau)}{h(s)}\right)^{\omega} \frac{h(s)^{-d}}{||\Phi_P(\tau, t_0, x_0)v_0||} d\tau \leq$$

$$\leq \frac{M}{m} \frac{h'(s)}{h(s)} \int_{s}^{s+1} \left(\frac{h(\tau)}{h(s)}\right)^{\omega+d} \frac{h'(\tau)}{h(\tau)} \frac{h(\tau)^{-d}}{||\Phi_P(\tau, t_0, x_0)v_0||} d\tau \leq$$

$$\leq \frac{M}{m} e^{H(\omega+d)} \frac{h'(s)}{h(s)} \int_{t_0}^{t} \frac{h'(\tau)}{h(\tau)} \frac{h(\tau)^{-d}}{||\Phi_P(\tau, t_0, x_0)v_0||} d\tau \leq$$

$$\leq \frac{DM}{m} e^{H(\omega+d)} \frac{h'(t)}{h(t)} \frac{h'(s)}{h(s)} \frac{h(t)^{-d}}{||\Phi_P(t, t_0, x_0)v_0||} \leq$$

$$\leq \frac{DMH}{m} e^{H(\omega+d)} \frac{h'(s)}{h(s)} \frac{h(t)^{-d}}{||\Phi_P(t, t_0, x_0)v_0||}.$$

Case 2. $t \in [s, s+1)$. It is similar to Case 2. from *Theorem 3.2* .

In a similar manner we obtain:

Case 1. $t \geq s+1$.

$$||\Phi_Q(s,t_0,x_0)v_0|| = \int_s^{s+1} ||\Phi_Q(s,t_0,x_0)v_0||d\tau \leq$$

$$\leq M \int_s^{s+1} \left(\frac{h(\tau)}{h(s)}\right)^{\omega} \frac{h'(\tau)}{h(\tau)} ||\Phi_Q(\tau,t_0,x_0)v_0||d\tau =$$

$$= M \int_s^{s+1} \left(\frac{h(t)}{h(s)}\right)^{-d} \left(\frac{h(\tau)}{h(s)}\right)^{\omega+d} \left(\frac{h(t)}{h(\tau)}\right)^{d} \frac{h'(\tau)}{h(\tau)} ||\Phi_Q(\tau,t_0,x_0)v_0||d\tau \leq$$

$$\leq Me^{H(\omega+d)} \left(\frac{h(t)}{h(s)}\right)^{-d} \int_{t_0}^{t} \left(\frac{h(t)}{h(\tau)}\right)^{d} \frac{h'(\tau)}{h(\tau)} ||\Phi_Q(\tau,t_0,x_0)v_0||d\tau \leq$$

$$\leq DMe^{H(\omega+d)} \left(\frac{h(t)}{h(s)}\right)^{-d} \frac{h'(t)}{h(t)} ||\Phi_Q(t,t_0,x_0)v_0||.$$

Case 2. $t \in [s, s+1)$. It is similar to Case 2. from *Theorem 3.2* .

□

Remark 3.4. If we consider in *Theorem 3.2* and *Theorem 3.3* the particular case when $h(t) = e^t$, then we obtain integral characterizations for the concept of uniform exponential dichotomy. Another direction for the study of these integral characterizations is given by Megan, Găină and Boruga (Toma) in [10] for the nonuniform case of dichotomy with differentiable growth rates, by Găină in [8] for uniform dichotomy with growth rates, by Boruga and Megan in [1] for uniform exponential dichotomy using evolution operators.

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THE ENERGY-CASIMIR MAPPING OF THE JERK VERSION OF AN ANHARMONIC OSCILLATOR

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Abstract

In this paper we study the stability and some special orbits of an one-parameter Hamilton-Poisson jerk system, namely the jerk version of an anharmonic oscillator. Moreover, we point out some properties of the energy-Casimir mapping associated to this system. [1]

1 Introduction

In a mechanical system, let $x = x(t)$ be the displacement of a moving object. Then, $\dot{x} = \dfrac{dx}{dt}$ and $\ddot{x} = \dfrac{d^2x}{dt^2}$ represents its velocity and acceleration, respectively. In addition, the jerk is the rate of change of acceleration, the third derivative $\dddot{x}$ of position with respect to time [7]. A differential equation of the form

$$\dddot{x} = j(x, \dot{x}, \ddot{x}),$$

where j is a smooth function usually, is called a jerk equation. This equation takes the form of a differential system, namely

$$\dot{x} = y, \dot{y} = z, \dot{z} = j(x, y, z),$$

called a jerk system.

The harmonic oscillator, described by the equation $\ddot{x} + \omega x^2 = 0$, models processes that exhibit sinusoidal oscillations with constant amplitude. Anharmonic oscillators are a type of oscillator that deviate from the simple harmonic motion. These oscillators are often described by equations that include nonlinear

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terms to account for the deviation from simple harmonic motion, and they can exhibit nonlinear behavior. Such an anharmonic oscillator is given by the equation $\ddot{x}+\delta x^n=0$ [4].

In this paper we study the jerk version of the anharmonic oscillator

$$\ddot{x}-\frac{g}{3}x^3=0,$$

namely the jerk equation

$$\dddot{x}-gx^2\dot{x}=0,$$

or equivalent, the jerk system

$$\begin{cases}\dot{x}=y\\ \dot{y}=z\\ \dot{z}=gx^2y\end{cases}, \tag{1}$$

where g is a real parameter.

The paper is organized as follows. In Section 2, we give the Hamilton-Poisson realization of the considered system. Then, following [8], we consider the energy-Casimir mapping associated to this realization. Moreover, we determine the image of this mapping. In Section 3, we point out some fibers of the energy-Casimir mapping in correspondence with some dynamical properties of the considered system, such as stable equilibrium points, periodic orbits, and so called split-homoclinic orbits.

2 Energy-Casimir mapping

In this section we point out a Hamilton-Poisson realization of the considered system and the image of the corresponding energy-Casimir mapping.

We consider the functions

$$H(x,y,z)=\frac{y^2}{2}-xz+\frac{gx^4}{4}\ ,\quad C(x,y,z)=z-\frac{gx^3}{3}. \tag{2}$$

Using (1) we deduce $\dot{H}=0$ and $\dot{C}=0$, thus H and C are constants of motion. Moreover, $\Pi\cdot\nabla C=\mathbf{0}$ and system (1) writes $\dot{\mathbf{x}}^T=\Pi\cdot\nabla H$, where $\mathbf{x}=(x,y,z)^T$ and

$$\Pi=\begin{bmatrix}0 & 1 & 0\\ -1 & 0 & -gx^2\\ 0 & gx^2 & 0\end{bmatrix}.$$

Consequently, system (1) has the Hamilton-Poisson realization $\left(\mathbb{R}^3,\Pi,H\right)$, where H is the Hamiltonian function and C is a Casimir of the Poisson structure given by Π.

In this framework, the energy-Casimir mapping associated to system (1) is defined by $\mathcal{EC}:\mathbb{R}^3\to\mathbb{R}^2$,

$$\mathcal{EC}(x,y,z)=(H(x,y,z),C(x,y,z))=\left(\frac{y^2}{2}-xz+\frac{gx^4}{4},z-\frac{gx^3}{3}\right). \tag{3}$$

We notice that system (1) writes $\dot{\mathbf{x}}^T=\nabla H\times\nabla C$. Therefore the critical points of energy Casimir mapping (3) are the equilibrium points of system (1), namely $E_M=(M,0,0), M\in\mathbb{R}$.

The image of the energy-Casimir mapping $\mathcal{EC}$ is the set

$$\mathrm{Im}(\mathcal{EC})=\left\{(h,c)\in\mathbb{R}^2|(\exists)(x,y,z)\in\mathbb{R}^3:\mathcal{EC}(x,y,z)=(h,c)\right\}.$$

As we have seen in other papers ([8]; also see [5] and references therein), the image $\mathcal{EC}(E_M)$ of the critical points through the energy-Casimir mapping leads to a partition of $\mathrm{Im}(\mathcal{EC})$. In our case, $\mathcal{EC}(E_M)$ is the curve

$$\Gamma=\left\{(h,c)\in\mathbb{R}^2|h=\frac{gM^4}{4},c=-\frac{gM^3}{3},M\in\mathbb{R}\right\}. \tag{4}$$

For $g\neq 0$, the curve Γ takes the form

$$\Gamma=\left\{(h,c)\in\mathbb{R}^2|c^4-\frac{64g}{81}h^3=0\right\}. \tag{5}$$

We define the sets

$$\Sigma_1=\left\{(h,c)\in\mathbb{R}^2|c^4-\frac{64g}{81}h^3<0\right\}, \tag{6}$$

$$\Sigma_2=\left\{(h,c)\in\mathbb{R}^2|c^4-\frac{64g}{81}h^3>0\right\}. \tag{7}$$

Proposition 2.1. *Let $g>0$ and let $\mathcal{EC}$ (3) be the energy-Casimir mapping of system (1). Then $Im(\mathcal{EC})=\Gamma\cup\Sigma_1\cup\Sigma_2=\mathbb{R}^2$ (Figure 2).*

Proof. A pair (h,c) belongs to the image of the energy-Casimir mapping if and only if the system

$$\begin{cases} h=\dfrac{y^2}{2}-xz+\dfrac{gx^4}{4}\\ c=z-\dfrac{gx^3}{3}\end{cases} \tag{8}$$

has at least a solution. Let us denote $h_e=\frac{gM^4}{4}$ and $c_e=-\frac{gM^3}{3}$, for any $M\in\mathbb{R}$.

Let $(h,c)\in\Gamma\cup\Sigma_1$ such that $h\geq h_e$. Using (8) we get

$$y^2=\frac{gx^4}{6}+2cx+2h,$$

or equivalent

$$y^2=\frac{g}{6}(x-M)^2[(x+M)^2+2M^2]+2(h-h_e).$$

Because $h \geq h_e$, system (8) has solution. Therefore, $\Gamma \cup \Sigma_1 \subset \mathrm{Im}(\mathcal{E}\mathcal{C})$.

Let $(h, c) \in \Sigma_2$ and $y = 0$. Using (8) we obtain

$$\frac{gx^4}{6} + 2cx + 2h = 0.$$

We will deduce that system (8) has solution by showing the above equation (2) has solution for $(h, c) \in \Sigma_2$. Consider the function $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \frac{gx^4}{6} + 2cx + 2h.$$

By studying the behavior of the function f, we have $f_{min} = 2h - \frac{3c}{2}\sqrt[3]{\frac{3c}{g}} < 0$ for any $(h, c) \in \Sigma_2$, hence the equation (2) has solution if $(h, c) \in \Sigma_2$. Therefore, system (8) has solution for $(h, c) \in \Sigma_2$. Thus $\Sigma_2 \subset \mathrm{Im}(\mathcal{E}\mathcal{C})$ and consequently $\mathrm{Im}(\mathcal{E}\mathcal{C}) = \mathbb{R}^2$, as required. □

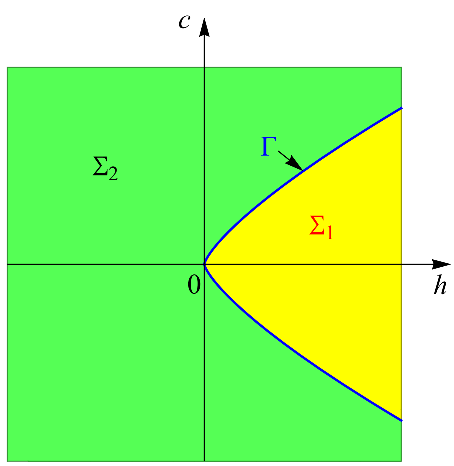

Figure 2: The image of the energy-Casimir mapping ($g > 0$).

Proposition 2.2. *Let* $g < 0$ *and let* $\mathcal{E}\mathcal{C}$ (3) *be the energy-Casimir mapping of system* (1)*. Then* $Im(\mathcal{E}\mathcal{C}) = \Gamma \cup \Sigma_2$ *(Figure 3).*

Proof. It is clear that system (8) has solution for $h = h_e = \frac{gM^4}{4}$ and $c = c_e = -\frac{gM^3}{3}$, for every $M \in \mathbb{R}$.

First, we prove that there are no pairs $(h, c) \in \mathrm{Im}(\mathcal{E}\mathcal{C})$ for $(h, c) \in \Sigma_1$ by showing that the system (8) has no solution. Suppose that system (8) has solution for $h < h_e$ and $c = c_e$. Using (8) we obtain

$$-g(x - M)^2[(x + M)^2 + 2M^2] + 6y^2 = 12\left(h - \frac{gM^4}{4}\right)$$

or equivalent

$$-g(x-M)^2[(x+M)^2+2M^2]+6y^2=12\,(h-h_e)\,. \tag{9}$$

But $h<h_e$, which leads to a contradiction. Thus, for $h<h_e$ and $c=c_e$ system (8) has no solution, that is $\Sigma_1\cap \mathrm{Im}(\mathcal{EC})=\emptyset$.

Now, let $(h,c)\in\Sigma_2$. Consider $y=0$ and the function f defined in (2). Using the similar method as in Proposition 2.1, we get that system (8) has solution for $(h,c)\in\Sigma_2$. Therefore $\Sigma_2\subset \mathrm{Im}(\mathcal{EC})$, and the conclusion follows. □

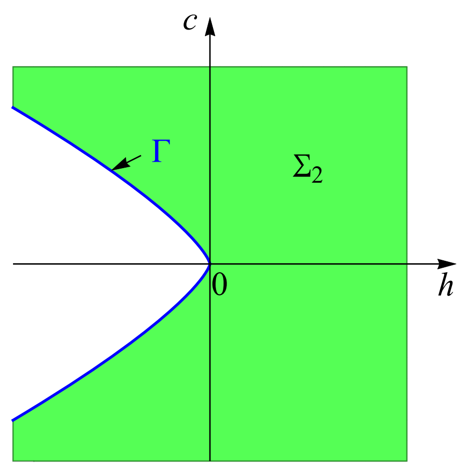

Figure 3: The image of the energy-Casimir mapping ($g<0$).

Proposition 2.3. *Let $g=0$ and let $\mathcal{EC}$ (3) be the energy-Casimir mapping of system* (1)*. Then $Im(\mathcal{EC})=\mathbb{R}^2\backslash\Sigma_0$, where $\Sigma_0=\{(h,c)\in\mathbb{R}^2|h<0, c=0\}$ (Figure 4).*

Proof. Let $(h,c)\in\Sigma_0$. Suppose that system (8) has solution for $h<0$ and $c=0$. System (8) becomes

$$\begin{cases} h=\dfrac{y^2}{2}-xz \\ z=0 \end{cases} \tag{10}$$

From (10) it follows that $h=\dfrac{y^2}{2}>0$ which leads to a contradiction. Thus, for $h<0$ and $c=0$ system (8) has no solution. Therefore $\Sigma_0\cap \mathrm{Im}(\mathcal{EC})=\emptyset$.

If $(h,c)\in\mathbb{R}^2\backslash\Sigma_0$, system (8) becomes

$$\begin{cases} h=\dfrac{y^2}{2}-xz \\ c=z \end{cases}\ . \tag{11}$$

Using (11) we get $cx+h=\dfrac{y^2}{2}>0$. There exist $x\in\mathbb{R}$ such that $cx+h>0$ for all $(h,c)\in\mathbb{R}^2\backslash\Sigma_0$, thus system (8) has solutions. Therefore, $\mathbb{R}^2\backslash\Sigma_0\subset \mathrm{Im}(\mathcal{E}\mathcal{C})$ and the conclusion follows. □

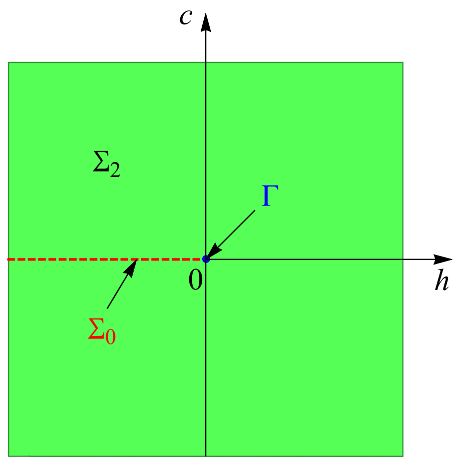

Figure 4: The image of the energy-Casimir mapping ($g=0$).

3 Connections between the energy-Casimir mapping and some dynamical properties of the system

In this section we study the stability of the equilibrium points, the existence of periodic orbits and other types of orbits of system (1) in connection with the image of the energy-Casimir mapping (3).

Proposition 3.1. *Let $g<0$. Then the equilibrium point $E_M=(M,0,0)$, $M\in\mathbb{R}$ is nonlinearly stable.*

Proof. The jacobian matrix of system (1) at E_M is

$$J(M,0,0)=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & gM^2 & 0\end{bmatrix}. \tag{12}$$

The corresponding eigenvalues are $\lambda_1=0, \lambda_{2,3}=\pm M\sqrt{g}$.

Let $M\neq 0$. Then the above Jacobian function (12) has a pair of imaginary eigenvalues. In this case we use the Arnold stability test [1]. We consider the function

$$F(x,y,z) = H(x,y,z) + MC(x,y,z). \tag{13}$$

The following hold:

1. $dF(M,0,0) = 0$.

2. $W = \ker dC(M,0,0) = \mathrm{span}_{\mathbb{R}}\{(1,0,gM^2),(0,1,0)\}$.

3. $d^2F(M,0,0)|_{W\times W} = dy^2 - \frac{1}{gM^2}dz^2$ is positive definite ($g<0$ and $M\neq 0$).

In conclusion, E_M is a nonlinear stable (Lyapunov stable) equilibrium point for $M \neq 0$ in the case $g<0$.

For $M=0$, the equilibrium point E_M becomes $(0,0,0)$. We show that the function $L \in C^\infty(\mathbb{R}^3,\mathbb{R})$,

$$L(x,y,z) = \left(\frac{y^2}{2} - xz + \frac{gx^4}{4}\right)^2 + \left(z - \frac{gx^3}{3}\right)^2 \tag{14}$$

is a Lyapunov function for system (1) and the equilibrium $(0,0,0)$.

It is easy to check that $\dot{L} = \nabla L \cdot \dot{\mathbf{x}} = 0$, where $\dot{\mathbf{x}} = (\dot{x},\dot{y},\dot{z})$ is given by (1). In addition, the condition $L(x,y,z)=0$ implies $\frac{y^2}{2} - xz + \frac{gx^4}{4} = 0$, $z - \frac{gx^3}{3} = 0$, whence $-gx^4 + 6y^2 = 0$. Because $g<0$, it follows that $x=y=0$ and consequently $z=0$. Thus $L(x,y,z)>0$, $\forall (x,y,z)\neq(0,0,0)$ and $L(0,0,0)=0$. In conclusion, L is a Lyapunov function and the equilibrium point $(0,0,0)$ is nonlinearly stable, which finishes the prove. □

Remark 3.2. *For $g<0$, the images of the stable equilibrium points $(M,0,0)$ through the energy-Casimir mapping* (3) *give the border* Γ (5) *of the set $Im(\mathcal{EC})$ (see Figure 3).*

The inverse image of $(h,c) \in \mathrm{Im}(\mathcal{EC})$ under the energy-Casimir mapping $\mathcal{EC}$ is the set

$$\mathcal{F}_{(h,c)} = \left\{(x,y,z)\in\mathbb{R}^3 \mid \mathcal{EC}(x,y,z) = (h,c)\right\}$$

called the fiber of $\mathcal{EC}$ corresponding to (h,c).

Proposition 3.3. *Let $g<0$ and $(h,c)\in\Gamma$ (4). Then the corresponding fiber $\mathcal{F}_{(h,c)}$ is a nonlinear stable equilibrium point, namely*

$$\mathcal{F}_{(h,c)} = \left\{(M,0,0)\,|\,M = -\sqrt[3]{\frac{3c}{g}}\right\}.$$

Proof. Let $(h,c) \in \Gamma$ (4). Then there is $M \in \mathbb{R}$ such that $h = \frac{gM^4}{4}$, $c = -\frac{gM^3}{3}$, namely $M = -\sqrt[3]{\frac{3c}{g}}$. The condition $\mathcal{EC}(x,y,z) = (h,c)$ is equivalent to

$$\begin{cases} \frac{y^2}{2} - xz + \frac{gx^4}{4} = \frac{gM^4}{4} \\ z - \frac{gx^3}{3} = -\frac{gM^3}{3} \end{cases}.$$

We get

$$-g(x-M)^2[(x+M)^2 + 2M^2] + 6y^2 = 0,$$

and taking into account that $g < 0$, it follows $x = M$, $y = z = 0$, as required. □

We expect that there are periodic orbits around each stable equilibrium point $(M, 0, 0)$. Indeed, we have.

Proposition 3.4. *Let $g < 0$ and $M \in \mathbb{R}^*$. Then for each sufficiently small $\varepsilon \in \mathbb{R}^*_+$, any integral surface*

$$\Sigma_\varepsilon^M : \frac{g}{12}(3x^4 - 4Mx^3 + M^4) + \frac{1}{2}y^2 - (x-M)z = \varepsilon^2$$

contains at least one periodic orbit γ_ε^M of system (1) whose period is close to $\frac{2\pi}{\omega}$, where $\omega = M\sqrt{-g}$.

Proof. The characteristic polynomial of the Jacobian matrix (12) of system (1) at the stable equilibrium point $E_M = (M,0,0)$ has the eigenvalues $\lambda_1 = 0$ and $\lambda_{2,3} = \pm iM\sqrt{-g}$. Hence we can apply a version of the Moser theorem regarding the existence of periodic orbits in the case of a zero eigenvalue [2].

The eigenspace corresponding to the eigenvalue zero, which is $\operatorname{span}_{\mathbb{R}}\{(1,0,0)\}$, has dimension 1.

Let us consider the constant of motion of system (1)

$$I(x,y,z) = \frac{y^2}{2} - xz + \frac{gx^4}{4} + M\left(z - \frac{gx^3}{3}\right). \tag{15}$$

It follows that:

1. $dI(M,0,0) = 0$.

2. $d^2I(M,0,0)|_{W\times W} = dy^2 - \frac{1}{gM^2}dz^2 > 0$ is positive definite $(g < 0, M \neq 0)$, where $W = \ker\, dC(M,0,0) = \operatorname{span}_{\mathbb{R}}\{(1,0,gM^2),(0,1,0)\}$.

Using the above-mentioned theorem, the conclusion follows. □

Remark 3.5. *Let $g<0$ and $(h,c)\in\Sigma_2$, i.e. (h,c) belongs to the interior of the image of $\mathcal{EC}$. Then the above proposition tells us that as long as the intersection of the level sets $H(x,y,z)=h, C(x,y,z)=c$ is a closed curve, the corresponding fiber $\mathcal{F}_{(h,c)}$ is a periodic orbit around the stable equilibrium point E_M (Figure 5). Moreover, when $\varepsilon\to 0$ these curves shrink to E_M. Such periodic orbits are presented in Figure 6, considering c fixed and h variable.*

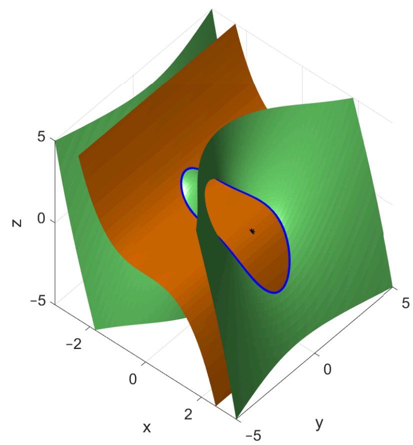

Figure 5: The fiber $\mathcal{F}_{(h,c)}$, $(h,c)\in\Sigma_2$: a periodic orbit.

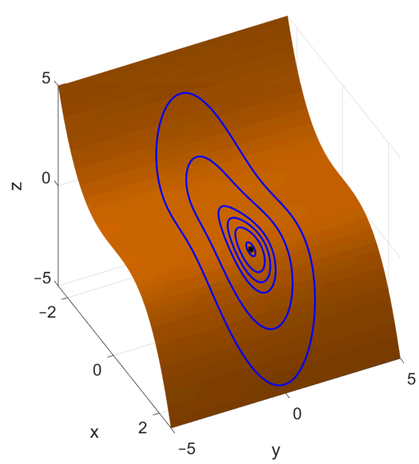

Figure 6: The family of periodic orbits around a stable equilibrium point (on the level set $C(x,y,z)=c$).

Proposition 3.6. *Let $g \geq 0$. Then the equilibrium point $E_M = (M, 0, 0)$, $M \in \mathbb{R}$ is unstable.*

Proof. If $g = 0$, system (1) becomes the equation $\dddot{x} = 0$, which has the unbounded solution $x(t) = \dfrac{C_1}{2}t^2 + C_2 t + C_3$, where $C_1, C_2, C_3 \in \mathbb{R}$. It results that E_M is an unstable equilibrium point.

Now, let $g > 0$. As we have seen, the eigenvalues of the jacobian matrix at E_M (12) are $\lambda_1 = 0, \lambda_{2,3} = \pm M\sqrt{g}$. Thus, if $M \neq 0$, then $\lambda_{2,3}$ is a pair of opposite sign real eigenvalues, thus E_M is unstable.

For $M = 0$, the eigenvalues the Jacobian function (12) are $\lambda_1 = \lambda_2 = \lambda_3 = 0$. In this case we determine a solution of system (1) that starts out near $(0, 0, 0)$, but it is unbounded, i.e. we prove that $(0, 0, 0)$ is unstable.

The dynamics of system (1) takes place at the intersection of the level sets $H(x, y, z) = constant, C(x, y, z) = constant$. We consider

$$H(x, y, z) = H(0, 0, 0), C(x, y, z) = C(0, 0, 0),$$

which is equivalent with

$$\begin{cases} \dfrac{y^2}{2} - xz + \dfrac{gx^4}{4} = 0 \\ z - \dfrac{gx^3}{3} = 0 \end{cases} . \tag{16}$$

We chose an initial condition (x_0, y_0, z_0) in a neighborhood of $(0, 0, 0)$ that satisfies (16), namely $x_0 = \varepsilon$, $y_0 = a\varepsilon^2$, $z_0 = 2a^2\varepsilon^3$, where $\varepsilon > 0$ is close to zero and $a = \sqrt{\dfrac{g}{6}}$. Then there is a solution $(x(t), y(t), z(t))$ with $(x(0), y(0), z(0)) = (x_0, y_0, z_0)$ which also satisfies (16). We get $y(t) = ax^2(t), z(t) = 2a^2x^3(t)$ and system (1) reduces in this case to the equation

$$\dot{x} = ax^2, a > 0.$$

We obtain the solution

$$x(t) = \frac{\varepsilon}{1 - a\varepsilon t}\ , \quad y(t) = \frac{a\varepsilon^2}{(1 - a\varepsilon t)^2}\ , \quad z(t) = \frac{2a^2\varepsilon^3}{(1 - a\varepsilon t)^3}, \quad t \in \left[0, \frac{1}{a\varepsilon}\right),$$

where $\varepsilon > 0$ is close to zero and $a = \sqrt{\dfrac{g}{6}}$.

It easy to see that this solution starts near equilibrium point $(0, 0, 0)$, but it does not stay near $(0, 0, 0)$. Therefore, the equilibrium point $(0, 0, 0)$ is unstable, which finishes the prove. □

For $g > 0$, the image of the energy-Casimir mapping is $\mathbb{R}^2$. The images of the unstable equilibrium points $(M, 0, 0)$ through $\mathcal{EC}$ belong to the curve Γ (5), and the corresponding fibers can contain homoclinic or heteroclinic orbits (see, e.g., [3]), or split-homoclinic and split-heteroclinic orbits [6]. In our case we have obtained split-homoclinic orbits.

Recall that a homoclinic orbit $\mathcal{H} : \mathbb{R} \to \mathbb{R}^3$ is a solution $(x(t), y(t), z(t))$ of the considered system which joins an unstable equilibrium point E to itself, that is $\mathcal{H}(t) := (x(t), y(t), z(t))$ and $\mathcal{H}(t) \to E$ as $t \to \pm\infty$. In addition, if $(x(t), y(t), z(t))$, $t \in (-\infty, s) \cup (s, \infty)$ is a solution of the considered system, which is indefinite in s, such that $(x(t), y(t), z(t)) \to E$ as $t \to \pm\infty$, then we say that $\mathcal{SH} : (-\infty, s) \cup (s, \infty) \to \mathbb{R}^3$, $\mathcal{SH}(t) := (x(t), y(t), z(t))$ is a split-homoclinic orbit.

Proposition 3.7. *Let $g > 0$, $(h, c) \in \Gamma, c \neq 0$, and $M = \sqrt[3]{\dfrac{-3c}{g}}$. Denote*

$$\begin{aligned}
x_1(t) &= \frac{M\left(e^{2p(t)} + 8e^{p(t)} - 2\right)}{e^{2p(t)} - 4e^{p(t)} - 2},\\
y_1(t) &= -\frac{12M^2 e^{p(t)}\sqrt{g}\left(e^{2p(t)} + 2\right)}{\left(e^{2p(t)} - 4e^{p(t)} - 2\right)^2},\\
z_1(t) &= \frac{gM^3}{3}\left(\frac{\left(e^{2p(t)} + 8e^{p(t)} - 2\right)^3}{\left(e^{2p(t)} - 4e^{p(t)} - 2\right)^3} - 1\right),
\end{aligned}$$

$$\begin{aligned}
x_2(t) &= \frac{M\left(2e^{2p(t)} - 8e^{p(t)} - 1\right)}{2e^{2p(t)} + 4e^{p(t)} - 1},\\
y_2(t) &= \frac{12M^2 e^{p(t)}\sqrt{g}\left(2e^{2p(t)} + 1\right)}{\left(2e^{2p(t)} + 4e^{p(t)} - 1\right)^2},\\
z_2(t) &= \frac{gM^3}{3}\left(\frac{\left(2e^{2p(t)} - 8e^{p(t)} - 1\right)^3}{\left(2e^{2p(t)} + 4e^{p(t)} - 1\right)^3} - 1\right),
\end{aligned}$$

with $p(t) = M(t\sqrt{g} + K)$ and $K \in \mathbb{R}$.

Then the fiber $\mathcal{F}_{(h,c)}$ contains two split-homoclinic orbits

$$\mathcal{SH}_1 : \mathbb{R} \setminus \left\{\frac{\ln\left(2 + \sqrt{6}\right) - KM}{M\sqrt{g}}\right\} \to \mathbb{R}^3, \mathcal{SH}_1 = (x_1, y_1, z_1),$$

$$\mathcal{SH}_2 : \mathbb{R} \setminus \left\{\frac{-KM - \ln(2) + \ln\left(\sqrt{6} - 2\right)}{M\sqrt{g}}\right\} \to \mathbb{R}^3, \mathcal{SH}_2 = (x_2, y_2, z_2),$$

which tend to the equilibrium point $E_M = (M, 0, 0)$ as $t \to \pm\infty$

Proof. The intersection of the level sets $H(x, y, z) = h, C(x, y, z) = c$ for $(h, c) \in \Gamma$ is given by

$$\begin{cases} H(x, y, z) = H(M, 0, 0) \\ C(x, y, z) = C(M, 0, 0) \end{cases},$$

or equivalent

$$\begin{cases} \dfrac{y^2}{2} - xz + \dfrac{gx^4}{4} = \dfrac{gM^4}{4} \\ z - \dfrac{gx^3}{3} = -\dfrac{gM^3}{3} \end{cases}.$$

We get

$$z = \frac{gx^3}{3} - \frac{gM^3}{3}, y^2 = \frac{1}{6}g(x-M)^2[(x+M)^2 + 2M^2]. \tag{17}$$

System (1) reduces in this case to the equation $\dot{x} = y$. First, we consider

$$\dot{x} = -\sqrt{\frac{g}{6}}(x-M)\sqrt{(x+M)^2 + 2M^2},$$

which is equivalent with

$$\int \frac{dx}{(x-M)\sqrt{(x+M)^2 + 2M^2}} = -\sqrt{\frac{g}{6}}t + K,\ K \in \mathbb{R}.$$

By integration, we obtain the solution x_1, and using (17), the functions y_1 and z_1, with $t \in \mathbb{R} \setminus \left\{ \dfrac{\ln\left(2+\sqrt{6}\right) - KM}{M\sqrt{g}} \right\}$.

Similarly, we reduce system (1) to

$$\dot{x} = \sqrt{\frac{g}{6}}(x-M)\sqrt{(x+M)^2 + 2M^2},$$

which is equivalent with

$$\int \frac{dx}{(x-M)\sqrt{(x+M)^2 + 2M^2}} = \sqrt{\frac{g}{6}}t + K,\ K \in \mathbb{R},$$

and we get the solution (x_2, y_2, z_2), $t \in \mathbb{R} \setminus \left\{ \dfrac{-KM - \ln(2) + \ln\left(\sqrt{6}-2\right)}{M\sqrt{g}} \right\}$.

We can also see that

$$\lim_{t\to\pm\infty}(x_1(t), y_1(t), z_1(t)) = (M, 0, 0),$$
$$\lim_{t\to\pm\infty}(x_2(t), y_2(t), z_2(t)) = (M, 0, 0),$$

as required. □

Remark 3.8. *In Figure 7, some fibers of the energy-Casimir mapping* (3) *are drawn on the level set $C(x, y, z) = c$, c fixed. More precisely, the fiber corresponding to $(h, c) = \mathcal{EC}(M, 0, 0) \in \Gamma$ contains a pair of split-homoclinic orbits (white curves). Keeping c fixed and changing h, we obtain unbounded orbits (yellow curves for $(h, c) \in \Sigma_1$ and green curves for $(h, c) \in \Sigma_2$).*

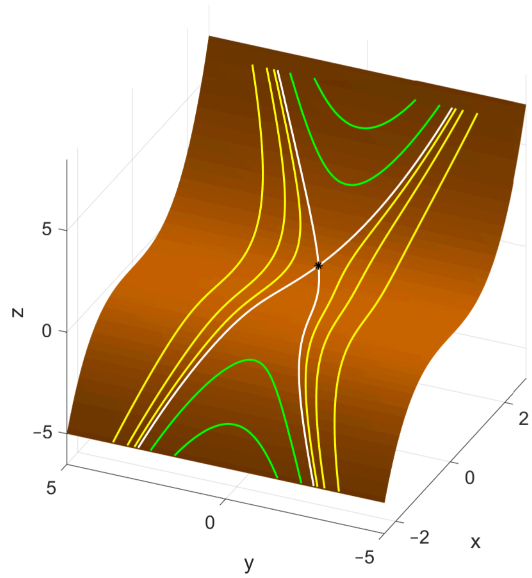

Figure 7: Split-homoclinic and other orbits near the unstable equilibrium point $E_M = (M, 0, 0)$, $M = \sqrt[3]{\dfrac{-3c}{g}}$, on the level set $C(x, y, z) = c$.

Proposition 3.9. *Let $g > 0$ and $x_0 \in \mathbb{R}^*$. Denote*

$$x_3(t) = \frac{x_0}{1 - ax_0 t}\ , \quad y_3(t) = \frac{ax_0^2}{(1 - ax_0 t)^2}\ , \quad z_3(t) = \frac{2a^2 x_0^3}{(1 - ax_0 t)^3},$$

$$x_4(t) = \frac{x_0}{1 + ax_0 t}\ , \quad y_4(t) = \frac{-ax_0^2}{(1 + ax_0 t)^2}\ , \quad z_4(t) = \frac{2a^2 x_0^3}{(1 + ax_0 t)^3},$$

where $a = \sqrt{\dfrac{g}{6}}$. Then the fiber $\mathcal{F}_{(0,0)}$ contains two split-homoclinic orbits

$$\mathcal{SH}_3 : \left(-\infty, \frac{1}{ax_0}\right) \cup \left(\frac{1}{ax_0}, \infty\right) \to \mathbb{R}^3, \quad \mathcal{SH}_3 = (x_3, y_3, z_3),$$

$$\mathcal{SH}_4 : \left(-\infty, -\frac{1}{ax_0}\right) \cup \left(-\frac{1}{ax_0}, +\infty\right) \to \mathbb{R}^3, \quad \mathcal{SH}_4 = (x_4, y_4, z_4),$$

which tend to the equilibrium point $E_0 = (0, 0, 0)$ as $t \to \pm\infty$ (Figure 8).

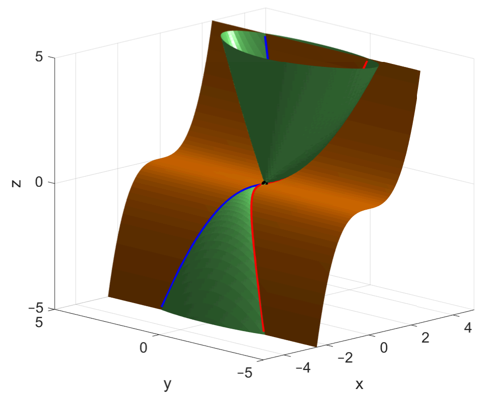

Figure 8: A pair of split-homoclinic orbits.

Proof. We consider the level sets

$$\begin{cases} H(x,y,z) = H(0,0,0) \\ C(x,y,z) = C(0,0,0) \end{cases},$$

that is

$$\begin{cases} \dfrac{y^2}{2} - xz + \dfrac{gx^4}{4} = 0 \\ z - \dfrac{gx^3}{3} = 0 \end{cases}.$$

Then

$$z = \frac{gx^3}{3}, y^2 = \frac{1}{6}x^4. \tag{18}$$

We chose an initial condition (x_0, y_0, z_0) that satisfies (18), that is $y_0 = ax_0^2$, $z_0 = 2a^2x_0^3$. System (1) reduces in this case to the equation

$$\dot{x} = ax^2, a > 0.$$

We obtain the solution x_3, and then y_3 and z_3, $t \in \mathbb{R}\backslash \left\{ \dfrac{1}{ax_0} \right\}$.

Similarly, choosing $y_0 = -ax_0^2$, $z_0 = 2a^2x_0^3$, we reduce system (1) to

$$\dot{x} = -ax^2, a > 0,$$

and we get the solution x_4, and then y_4 and z_4, $t \in \mathbb{R}\backslash \left\{ -\dfrac{1}{ax_0} \right\}$.

We can also see that

$$\lim_{t\to\pm\infty} (x_3(t), y_3(t), z_3(t)) = (0,0,0),$$
$$\lim_{t\to\pm\infty} (x_4(t), y_4(t), z_4(t)) = (0,0,0),$$

as required. $\square$

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THE DETERMINATION OF THE STATISTICAL DISTRIBUTION FUNCTION IN THE TESTING OF PRODUCT LIFE

Marius Valentin BOLDEA

Abstract

In this paper I consider that a series type system is a system that functions only if all its components are functioning. In case that the distribution functions for all the components of a series type system are known, then the distribution function of the whole system is given by relation (9). Particular cases are analysed when the components are exponential distributions of the Weibull or Rayleigh type. [1]

The theory of durability has found large application in the statistical control of products, in the security of the technical systems, as well as in demographic problems such as those referring to the medium duration of life etc.

The above mentioned theory has been dealt with in several works. One must mention in this regard the basic works written by B. Epstein and M. Sobel. Control plans have been elaborated, being based on different rules of statistical distribution such as: exponential [2], Rayleigh [3], Weibull [4], gamma [5].

It follows that one of the outstanding problems of the theory of durability is the determination of the function of distribution. Although there are sufficient possibilities to determine a rule of distribution, the former are insufficient and their approximation to reality is rather unsatisfactory. For this reason we suggest new functions of distribution in this domain, even if some of them are combined with those existent.

Since $f(t)$ is the function of distribution in time, the function of distribution will be:

$$F(t) = \int_0^t f(u)du. \tag{1}$$

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The function $P(t)$ called function of survival (in demography), respectively function of security or fiability (in technics) is defined:

$$P(t) = 1 - F(t). \tag{2}$$

The mortality rate, respectively the rate of destruction is defined:

$$\lambda(t) = f(t)/(1 - F(t)). \tag{3}$$

The following relations are to be observed between these three functions:

$$f(t) = \lambda(t)P(t) \tag{4}$$

$$P(t) = e^{-\int_0^t \lambda(u)du} \tag{5}$$

The testing of the durability of a complex system cannot be carried out without a previous analysis of the durability of the component parts of the system. The question that arises is to determine the same function when the function of distribution of the component parts is known.

A system usually functions only if each component is functioning i. e. a series type connection. In these conditions the probability of their functioning is given by the rule of the multiplication of the probability of independent cases. In case that $f_i(t)$, $(i = 1, 2, ..., n)$ are the rules of the distribution of the elements of the systems, then $\lambda_i(t)$ and $P_i(t)$ are the rates, respectively the corresponding security functions.

Since $P_i(t)$ is the probability that the component i should not be out of order up to the moment t, the probability of the functioning of the whole system is:

$$P_s(t) = P_1(t) \cdot P_2(t) \cdot ... \cdot P_n(t) \tag{6}$$

Because of relation (5), relation (6) becomes:

$$P(t) = e^{-\int_0^t \sum_{i=1}^n \lambda_i(u)du} \tag{7}$$

We note: $\lambda_s = \sum_{i=1}^n \lambda_i$ and relation (7) becomes:

$$P_s(t) = e^{-\int_0^t \lambda_s du} \tag{8}$$

Also:

$$f_s(t) = \lambda_s \cdot P_s = \sum_{i=1}^n \lambda_i \cdot \prod_{i=1}^n P_i = \sum_{i=1}^n \lambda_i \cdot \frac{\prod_{i=1}^n f_i}{\prod_{i=1}^n \lambda_i}$$

and

$$f_s(t) = \frac{\sum_{i=1}^n \lambda_i}{\prod_{i=1}^n \lambda_i} \cdot \prod_{i=1}^n f_i \tag{9}$$

If the functions of distributions for the components of the systems are $f_i(t)$ the function of distribution of the whole system is given by relation (9).

We may conclude that in a series type system the rates are additive.

We consider that the functions of all the components of the system follow Weibull's rule:

$$f_i(t) = k \cdot a_i \cdot t^{k-1} \cdot e^{-a_i \cdot t^k}$$

with the rate:

$$\lambda_i(t) = k \cdot a_i \cdot t^{k-1}$$

Calculating the functions of distribution of the system with relation (9) it results that:

$$f_s(t) = \frac{k \cdot \sum_{i=1}^{n} a_i \cdot t^{k-1}}{k^n \cdot (t^{k-1})^n \cdot \prod_{i=1}^{n} a_i} \cdot k^n \cdot (t^{k-1})^n \cdot \prod_{i=1}^{n} a_i \cdot e^{-\sum_{i=1}^{n} a_i t^k}$$

so

$$f_s(t) = k \cdot \sum_{i=1}^{n} a_i \cdot t^{k-1} \cdot e^{-\sum_{i=1}^{n} a_i t^k} \tag{10}$$

which is a Weibull distribution too.

Thus, if in a series type system each component follows a rule of Weibull distribution, the system itself functions according to a rule of Weibull distribution.

Special cases: let be $k = 1$, Weibull's rule is the exponential distribution and the theorem becomes: if in a series type system each component follows a rule of exponential distribution, the system also follows a rule of exponential distribution.

Let be $k = 2$, Weibuul's rule is the Rayleigh distribution and the theorem becomes: if in a series type system each component follows a Rayleigh distribution rule, the system also follows a Rayleigh distribution rule.

Thus one can explain the occurrence of systems with distributions of the exponential or Rayleigh type.

Let us suppose that each component of the system follows another rule of distribution, but each of them of the Weibull type, i. e.:

$$f_i(t) = k_i \cdot a_i \cdot t^{k_i-1} \cdot e^{-a_i \cdot t^{k_i}}$$

Using relation (9) we get the distribution of system:

$$f_s(t) = \sum_{i=1}^{n} k_i \cdot a_i \cdot t^{k_i-1} \cdot e^{-\sum_{i=1}^{n} a_i \cdot t^{k_i}} \tag{11}$$

which is no longer a Weibull distribution.

For example, a system consisting of two subsystems, one of them with an exponential distribution rule and the other with a Rayleigh type rule, will have a distribution rule got from the relation (11) for $k_1 = 1$ and $k_2 = 2$ as follows:

$$f_s(t) = (a_1 + 2a_2 \cdot t) \cdot e^{-(a_1 \cdot t + a_2 \cdot t^2)} \tag{12}$$

If only the experimental data of the system are known, to determine its distribution function, it is necessary to calculate and to represent graphically the rate of distribution from the experimental data in the histogram. Then the function of this curve is to be determined, preferably a sum of other functions as simple as possible, if possible a sum of exponentials, admitting that the system consisted of Weibuul components.

Provided that the rate is a sum of exponentials i. e.:

$$\lambda_s(t) = \sum_{i=1}^{n} c_i \cdot t^{b_i}$$

it may be formulated as follows:

$$\lambda_s(t) = \sum_{i=1}^{n} k_i a_i \cdot t^{k_i - 1}$$

and the function of distribution is given by relation (11).

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