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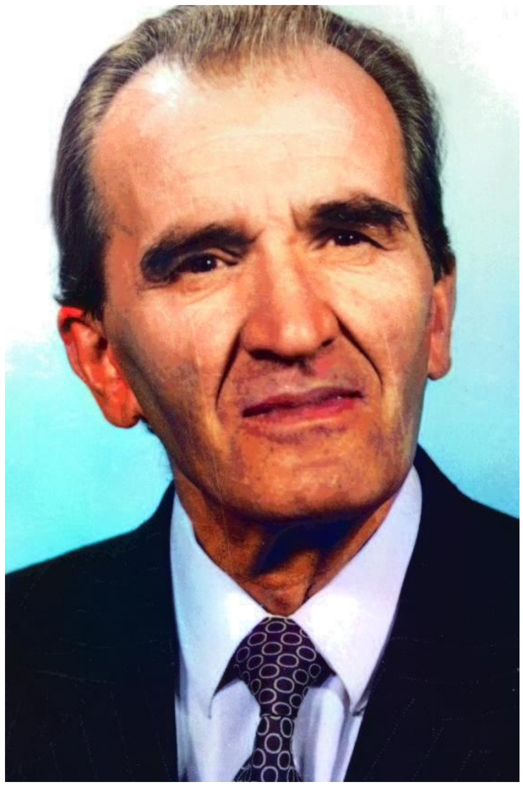
Contents

Dedicated to the memory of Prof. dr. Borislav D. CRSTICI
(26 March 1924 - 20 August 2014)



Borislav D. CRSTICI (Borislav D. KRSTIĆ, to friends - Bora) was born on March 26, 1924, in the village of Belobreșca, Caraș-Severin County. His father, Dușan Crstici, a great lover of the written word, worked as the chief postal officer in the locality, while his mother, Desanca, born Stanoievici, in the locality of Crușcița (Serbia), near the Romanian-Serbian border, was a homemaker. Borislav had two brothers: Cedomir, who later became a doctor, and Petru, who disappeared during World War II. From his marriage to Jelka Crstici, born Miatov, from Cenei, Timiș County, a physics teacher, they had two children: son Dușan, an oncologist, and daughter Desanca, a computer scientist, both born in Timișoara. Borislav D. Crstici passed away on August 20, 2014, and was buried in Timișoara, in the cemetery on Stuparilor Street.

Regarding his education and professional training, Borislav-Bora attended elementary school in the locality of Sușca, close to his native Belobreșca (1931-1935), and began his gymnasium studies at the Serbian section of the Gymnasium belonging to the "C. D. Loga" High School in Timișoara (1935-1936), continuing, due to the war, at the Gymnasium in Petrograd (now the city of Zrenjanin, Serbia), where he also completed his high school studies, which he could finalize only towards the end of the war, in 1945, taking the baccalaureate exam in the city of Osijek (now in Croatia). He pursued his university studies at the Faculty of Mathematics and Physics at "Babeș-Bolyai" University in Cluj (1946-1950). Being an exceptionally prepared student, towards the end of his studies (1949-1950), he was co-opted as a trainee assistant at the Mathematics Department of the faculty. However, the longing for his native lands led him, after graduation, to give up the position of university assistant to return to Banat. Here, he took a job as a mathematics teacher at the mixed Serbian High School in Timișoara (1950-1954). Nevertheless, his attraction to mathematical research made him work, in parallel, as an hourly-paid assistant at the Mathematics Department of the Polytechnic Institute in Timișoara. After 1954, he became fully engaged at the Polytechnic, where he remained active throughout his life, even after retirement. His teaching career followed an upward course: university assistant (1951-1959), lecturer (1959-1964), associate professor (1964-1971), and university professor (1971-1986), until his retirement (September 10, 1986), after which he continued to work for a few years as a consulting professor (1986-1991). During this period, he served as the head of the (unique) Mathematics Department of the "Traian Vuia" Polytechnic Institute in Timișoara (1971-1972, 1972-1979), being also the only doctoral supervisor in the field of mathematics at this higher institute (1977-1990).

He obtained his doctorate in mathematics in 1969, at the Faculty of Mathematics and Mechanics of "Babeș-Bolyai" University in Cluj. His doctoral thesis, developed under the supervision of Prof. Dr. Tiberiu Popoviciu, is titled On Functional Equations Defining Polynomials and comprises 181 pages.

As a university teacher, over the years, Borislav Crstici taught students (mainly from the Faculty of Electrical Engineering) the subjects: Analytical and Differential Geometry, Special Mathematics. Regarding his scientific research activity, he was mainly concerned with the field of functional equations with multiple variables, as well as number theory. This resulted in dozens of articles published in journals and proceedings both nationally and internationally, scientific monographs at prestigious foreign publishers, as well as educational manuals and problem collections.

Prof. Dr. Borislav Crstici also had a rich international scientific collaboration, being a pioneer, considering the political restrictions of that time. He facilitated the establishment of scientific connections with prestigious European universities. Notably, he co-authored books published by prestigious foreign publishers: József Sándor, Dragoslav Mitrinović, Borislav Crstici, Handbook of Number Theory I, ed. I, Kluwer Acad. Publ., Dordrecht, 1995; ed. II, Springer, 2006; Jozsef Sandor and Borislav Crstici, Handbook of Number Theory II, Kluwer Acad. Publ., Dordrecht, 2004. As a recognition of his scientific activity, he was elected a member of several scientific societies: Member of the Romanian Mathematical Society (1951–1991); Member of the "American Mathematical Society" (since 1975);

reviewer for international mathematical journals: "Mathematical Reviews" and "Zentralblatt für Mathematik" (since 1965).

We cannot overlook the awards he received: the Ministry of Education and Research Prize for the book by Borislav Crstici, Octavian E. M. Gheorghiu, Analytical and Differential Geometry, Didactic and Pedagogical Publishing House, Bucharest, 1968; the "Cultural Merit" diploma (onthe occasion of the 50th anniversary of the Polytechnic School in Timișoara, 1970). Additionally, he was the editor-in-chief of the publication "Scientific Bulletin of the 'Politehnica' University of Timișoara, Transactions on Mathematics and Physics," from 1978 to 2014. He presided over the works of the "Tiberiu Popoviciu" Seminar on Functional Equations, Approximation, and Convexity, while also being the editor of the respective proceedings. He participated in organizing the works of the "Symposium of Mathematics and its Applications" at the "Traian Vuia" Polytechnic Institute in Timișoara (1985). Together with his wife, Prof. Jelka Crstici, he translated mathematics manuals from Romanian into Serbian, necessary for students in schools teaching in Serbian in Romania, and similarly, translated mathematics manuals from Serbian into Romanian for the needs of schools teaching in Romanian in Vojvodina (Serbia). An important place in his translation activity is occupied by the translation into Serbian, for the Nolit Publishing House in Belgrade, of the book by Solomon Marcus – Mathematical Poetics, Ed. of the Romanian Academy (in collaboration with Prof. Dr. Dragan Stoianovici, from the University of Bucharest).

In addition to publications in the field of mathematics, Prof. Dr. Borislav Crstici published many essays on cultural, literary, philosophical, and historical themes in Serbian newspapers and literary magazines in Romania. In conclusion, we can say that the entire life and scientific and social activity of Prof. Dr. Borislav Crstici unfolded under the sign of the Latin maxim that guided his entire life: "Labor omnia vincit improbus" (Virgil, "Georgica," 1, 145-146).

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In memory of Professor Borislav D. CRSTICI on the Centenary of his birth.

STRONG DICHOTOMY WITH GROWTH RATES FOR LINEAR DISCRETE TIME SYSTEMS IN BANACH SPACES

Carmen-Florinela POPA

Abstract

The purpose of the this paper is to present characterizations of Datko-type for the concepts of strong h-dichotomy and h-dichotomy, are given using both invariant and strongly invariant projections sequences, for linear discrete time systems in Banach spaces. Also, as consequences we obtain characterizations for uniform h-dichotomy, strong eponential/polinomyal dichotomy and exponential/polynomial dichotomy. [1]

1 Introduction

Nowadays, it is required a discrete-time approach because the phenomena of the real world, in different domain takes place in a certain moment of time. The notion of exponential dichotomy introduced by Perron for differential equations and by Li for difference equations an dynamical systems.

In some situations, particularly in the nonautonomous settings, the concept of uniform exponential dichotomy is too restrictive and it is important to consider more general behaviors. One of the main reasons for weakening the assumption of uniform exponential dichotomy is that from the point of view of ergodic theory almost all variational equations in a finite-dimensional space admit a nonuniform

[1]Mathematical Subject Classification (2010): 34D05; 34D09; 37B55; 93C05; 93C55
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exponential dichotomy. On the other hand it is important to treat the case of noninvertible systems because of their interest in applications (e.g., random dynamical systems, generated by random parabolic equations, are not invertible). Lately, characterizations of the nonuniform exponential dichotomy for discrete linear systems can be found in the works [3], [23], [28], [27] [26], [18], [19], [21], [30]. Characterizations of the nonuniform exponential dichotomy for discrete linear systems can be found in the works [5], [25], [22],[6], [18], [20].
Later, the concept of polynomial dichotomies was studied by Boruga, Seimeanu, Crai, Dragicevic and Sasu, Yue in works [7], [11], [13], [14], [29], [31].
The notion of dichotomy with growth rates in discrete-time was studied in [1], [8], [9].
In this paper we obtain different characterizations of Datko type for the dichotomy and strong dichotomy with growth rates for linear discrete time systems in Banach spaces with respect to invariant and strongly invariant projections sequences.

2 Preliminaries

Let X be a real or complex Banach space and $\mathcal{B}(X)$ the Banach algebra of all bounded operators from X into itself. The norms of both these spaces will be denote by $||\cdot||$. Let $\mathbb{N}$ be the set of all positive intergers and we deonte by Δ and T the follwing sets

$\Delta = \{(m,n) \in \mathbb{N}^2 : m \geq n\} \quad T = \{(m,n,p) \in \mathbb{N}^3 : m \geq n \geq p\}$.

In this paper we consider linear discrete-time systems of the form

$$(\mathcal{A}) \qquad x_{n+1} = A_n x_n, \quad n \in \mathbb{N}$$

where (A_n) is a sequence in $\mathcal{B}(X)$.
Then every solution $x = (x_n)$ of system $(\mathcal{A})$ is given by

$$x_m = A_m^n x_n, \quad \textit{for all } (m,n) \in \Delta,$$

where

$$A_m^n = \begin{cases} A_{m-1}A_{m-2}...A_n, & m > n \\ I, & m = n \end{cases}$$

and I is the identity operator on X.

Remark 2.1. *We have the following properties:*
(i) $A_{n+1}^n = A_n$, *for all* $n \in \mathbb{N}$
(ii) $A_m^n A_n^p = A_m^p$, *for all* $(m,n,p) \in T$.

Definition 2.2. *A nondecreasing sequence* (h_n) *on* $[1,\infty)$ *is called growth rate sequence if* $\lim_{n\to\infty} h_n = \infty$.

Definition 2.3. *A sequence (P_n) on $\mathcal{B}(X)$ is called projections sequence on X if*

$$P_n^2 = P_n, \quad for\ all\ n \in \mathbb{N}$$

Remark 2.4. *If (P_n) is projections sequence on X, then the sequence $Q_n = I - P_n$ is also a projections sequence on X, called the complementary projections sequence of P_n with $KerQ_n = RangeP_n$ and $RangeQ_n = KerP_n$ for every $n \in \mathbb{N}$.*

Definition 2.5. *The sequence (P_n) is invariant for the linear system $(\mathcal{A})$, if*

$$A_n P_n = P_{n+1} A_n, \quad for\ all\ n \in \mathbb{N}$$

Remark 2.6. *If (P_n) is invariant for $(\mathcal{A})$ then*

$$A_m^n P_n = P_m A_m^n \quad A_m^n Q_n = Q_m A_m^n,$$

for all $(m, n) \in \Delta$.

Definition 2.7. *The sequence (P_n) is strongly invariant for the linear system $(\mathcal{A})$ if it is invariant for $(\mathcal{A})$ and the restriction of A_m^n is an isomorphism from Range Q_n to Range Q_m.*

Remark 2.8. *If the sequence of projections (P_n) is a strongly invariant for the system $(\mathcal{A})$, exists $B : \Delta \to \mathbb{R}$ with $B(n, m) = B_n^m$: Range $Q_m \to$ Range Q_n isomorphism with $A_m^n B_n^m Q_m = Q_m$ and $B_n^m Am^n Q_n = Q_n$.*

Remark 2.9. *If (P_n) is a strongly invariant projections sequence for $(\mathcal{A})$ then there exists $B(n, m)$ as in the Remark 1.2.4 with the following properties:*

1. $B_n^m Q_m = Q_n B_n^m Q_m$
2. $Q_m = B_m^m Q_m = Q_m B_m^m Q_m$
3. $B_p^m Q_m = B_p^n B_n^m Q_m$

for all $(m, n), (n, p) \in \Delta$.

Proof. 1. For all $(m, n, x) \in \Delta \times X$ we have

$$Q_m x \in ImQ_m = KerP_m$$

so

$$B_m^n Q_m x \in KerP_n x = ImQ_m$$

then we have

$$B_n^m Q_m = B_n^m A_m^n B_n^m Q_m =$$

$$= B_n^m A_m^n Q_n B_n^m Q_m = Q_n B_n^m Q_m.$$

2. From Remark 4 we have that

$$Q_m = A_m^n B_n^m Q_m$$

For $n = m$ we have
$$A_m^m B_m^m Q_m = B_m^m Q_m = Q_m.$$

3.
$$B_p^n B_n^m Q_m = B_p^n B_n^m A_m^n B_n^m Q_m =$$
$$= B_p^n B_n^m A_m^n A_n^p Q_n B_n^m Q_m = B_p^n B_n^m A_m^n Q_n A_n^p B_n^m Q_m =$$
$$= B_p^n Q_n A_n^p B_n^m Q_m = B_p^n A_n^p Q_n B_n^m Q_m =$$
$$= Q_n B_n^m Q_m = B_n^m Q_m.$$

□

3 The growth concepts for dichotomy with growth rates

Definition 3.1. *If the pair $(\mathcal{A}, P)$ has h-growth, then there are $M \geq 1, \omega > 0$ and $\delta \geq 0$ then we have:*

(hg_1) $h_n^\omega ||A_m^p P_p x|| \leq M h_m^\omega \dfrac{h_{n+1}}{h_n} h_n^\delta ||A_n^p P_p x||$

(hg_2) $h_n^\omega ||A_n^p Q_p x|| \leq M h_m^\omega \dfrac{h_{m+1}}{h_m} h_m^\delta ||A_m^p Q_p x||$

for all $(m, n, p, x) \in T \times X$.

Remark 3.2. *The pair $(\mathcal{A}, P)$ has h-growth if and only if there are $M \geq 1, \omega > 0$ and $\delta \geq 0$ with:*

(hg_1') $h_n^\omega ||A_m^n P_n x|| \leq M h_m^\omega \dfrac{h_{n+1}}{h_n} h_n^\delta ||P_n x||$

(hg_2') $h_n^\omega ||Q_n x|| \leq M h_m^\omega \dfrac{h_{m+1}}{h_m} h_m^\delta ||A_m^n Q_n x||$

for all $(m, n, x) \in \Delta \times X$.

Proof. Necessity. Results from Definition 5 for $p = n$.

Sufficiency. For $(hg_1') \implies (hg_1)$ we have

$$||A_m^p P_p x|| = ||A_m^n A_n^p P_p x|| \leq M \left(\frac{h_m}{h_n}\right)^\omega h_n^\delta \frac{h_{n+1}}{h_n} ||A_n^p P_p x||$$

for all $(m, n, p, x) \in T \times X$.

For $(hg_2') \implies (hg_2)$ we have

$$||A_m^p Q_p x|| = ||A_m^n A_n^p Q_p x|| \geq \frac{1}{M} \left(\frac{h_n}{h_m}\right)^\omega h_m^{-\delta} \frac{h_m}{h_{m+1}} ||A_n^p Q_p x||$$

for all $(m, n, p) \in T \times X$. □

Remark 3.3. *The particular cases for the concept of h-growth are:*

1. *If $\delta = 0$, then we obtain the concept of uniform h-growth.*
2. *If $h_m = e^m$, we have exponential growth.*

Definition 3.4. *If the pair $(\mathcal{A}, P)$ has strong h-growth (s.h.g) then there are $M \geq 1, \omega > 0$ and $\delta \geq 0$ such that:*

(shg_1) $h_n^\omega ||A_m^p P_p x|| \leq M h_m^\omega h_n^\delta ||A_n^p P_p x||$

(shg_2) $h_n^\omega ||A_n^p Q_p x|| \leq M h_m^\omega h_m^\delta ||A_m^p Q_p x||$

for all $(m, n, p, x) \in \Delta \times X$.

Remark 3.5. *The pair $(\mathcal{A}, P)$ has strong h-growth if and only if there are $M \geq 1$, $\omega > 0$ and $\delta \geq 0$ such that:*

$(shg_1^{'})$ $h_n^\omega ||A_m^n P_n x|| \leq M h_m^\omega h_n^\delta ||P_n x||$

$(shg_2^{'})$ $h_n^\omega ||Q_n x|| \leq M h_m^\omega h_m^\delta ||A_m^n Q_n x||$

for all $(m, n, x) \in \Delta \times X$.

Proof. Necessity: Results from Definition 6 for $p = n$.

Sufficiency: For $(shg_1') \implies (shg_1)$ we have

$$h_n^\omega ||A_m^p P_p x|| = h_n^\omega ||A_m^n A_n^p P_p x|| \leq$$

$$\leq M h_m^\omega h_n^\delta ||A_n^p P_p x||$$

and for $(shg_2') \implies (shg_2)$ we have

$$||A_m^p Q_p x|| = ||A_m^n A_n^p Q_p x|| \geq \frac{1}{M} \left(\frac{h_n}{h_m} \right)^\omega h_m^{-\delta} ||A_n^p Q_p x||$$

for all $(m, n, p) \in T \times X$. $\square$

Remark 3.6. *The particular cases for the concept of strong h-growth are:*

1. *If $\delta = 0$, then we obtain the concept of strong uniform h-growth.*
2. *If $h_m = e^m$, we have strong exponential growth.*

Proposition 3.7. *Let (P_n) strong invariant. The pair $(\mathcal{A}, P)$ has strong h-growth if and only if there are $M \geq 1, \omega > 0$ and $\delta \geq 0$ with:*

$(hg_1^{s''})$ $h_n^\omega ||A_m^n P_n x|| \leq M h_m^\omega h_n^\delta ||P_n x||$

$(hg_2^{s''})$ $h_n^\omega ||B_n^m Q_m x|| \leq M h_m^\omega h_m^\delta ||Q_m x||$

for all $(m, n, x) \in \Delta \times X$.

Proof. Neccessity. The implication $(hg_1^{'}) \implies (hg_1^{s'})$ is immediate for Remark 6. We will to prove $(hg_2^{'}) \implies (hg_2^{s'})$. From $(hg_2^{'})$ and Remark 5 we have

$$h_n^\omega ||Q_n B_n^m Q_m x|| = h_n^\omega ||Q_n B_n^m Q_m x|| \leq$$

$$\leq M h_m^\omega h_m^\delta ||A_m^n Q_n B_n^m Q_m x|| \leq M h_m^\omega h_m^\delta ||Q_m A_m^n B_n^m Q_m x|| \leq$$

$$\leq Mh_m^{\omega} h_m^{\delta} ||Q_m x||$$

for all $(m, n, x) \in \Delta \times X$.

Sufficiency. The implication $(hg_1^{s'}) \implies (hg_1^{'})$ is immediate for Remark 6. We will to prove $(hg_2^{s'}) \implies (hg_2^{'})$. From $(hg_2^{s'})$ and Remark 5 we have

$$h_n^{\omega}||Q_n x|| = h_n^{\omega}||B_n^m A_m^n Q_n x|| \leq Mh_m^{\omega} h_m^{\delta}||A_m^n Q_n x||$$

for all $(m, n, x) \in \Delta \times X$. □

Proposition 3.8. *e Let (P_n) is strong invariant, the pair $(\mathcal{A}, P)$ has h-growth if and only if there are $M \geq 1, \omega > 0$ and $\delta \geq 0$ with:*

$(shg_1^{s'})$ $h_n^{\omega}||A_m^n P_n x|| \leq Mh_m^{\omega} \dfrac{h_{n+1}}{h_n} h_n^{\delta}||P_n x||$

$(shg_2^{s'})$ $h_n^{\omega}||B_n^m Q_m x|| \leq Mh_m^{\omega} \dfrac{h_{m+1}}{h_m} h_m^{\delta}||Q_m x||$

for all $(m, n, x) \in \Delta \times X$.

Proof. Neccessity. The implication $(shg_1^{'}) \implies (shg_1^{s'})$ is immediate for Remark 8. We will to prove $(shg_2^{'}) \implies (shg_2^{s'})$. From $(shg_2^{'})$ and Remark 5 we have

$$h_n^{\omega}||Q_n B_n^m Q_m x|| = h_n^{\omega}||Q_n B_n^m Q_m x|| \leq$$

$$\leq Mh_m^{\omega} h_m^{\delta} \frac{h_{m+1}}{h_m}||A_m^n Q_n B_n^m Q_m x|| \leq Mh_m^{\omega} h_m^{\delta} \frac{h_{m+1}}{h_m}||Q_m A_m^n B_n^m Q_m x|| \leq$$

$$\leq Mh_m^{\omega} h_m^{\delta} \frac{h_{m+1}}{h_m}||Q_m x||$$

for all $(m, n, x) \in \Delta \times X$.

Sufficiency. The implication $(shg_1^{s'}) \implies (shg_1^{'})$ is immediate for Remark 8. We will to prove $(shg_2^{s'}) \implies (shg_2^{'})$. From $(shg_2^{s'})$ and Remark 4 we have

$$h_n^{\omega}||Q_n x|| = h_n^{\omega}||B_n^m A_m^n Q_n x|| \leq Mh_m^{\omega} h_m^{\delta} \frac{h_{m+1}}{h_m}||A_m^n Q_n x||$$

for all $(m, n, x) \in \Delta \times X$. □

4 The concepts of h-dichotomy with growth rates

Definition 4.1. *If then the pair $(\mathcal{A}, P)$ is h-dichotomic (h.d.) then there are $N \geq 1, \epsilon \geq 0$ and $\nu > 0$ such that:*

(hd$_1$) $h_m^{\nu}||A_m^p P_p x|| \leq Nh_n^{\nu} h_n^{\epsilon} \dfrac{h_{n+1}}{h_n}||A_n^p P_p x||$

(hd$_2$) $h_m^{\nu}||A_n^p Q_p x|| \leq Nh_n^{\nu} h_m^{\epsilon} \dfrac{h_{m+1}}{h_m}||A_m^p Q_p x||$

for all $(m, n, p, x) \in T \times X$.

Remark 4.2. *The pair (A, P) is h-dichotomic if and only if there are $N \geq 1$, $\nu > 0$ and $\epsilon \geq 0$ such that:*

$(hd_1^{'})$ $h_m^\nu ||A_m^n P_n x|| \leq N h_n^\nu h_n^\epsilon \dfrac{h_{n+1}}{h_n} ||P_n x||$

$(hd_2^{'})$ $h_m^\nu ||Q_n x|| \leq N h_n^\nu h_m^\epsilon \dfrac{h_{m+1}}{h_m} ||A_m^n Q_n x||$

for all $(m, n, x) \in \Delta \times X$.

Proof. Necessity. Results from Definition 7 for $p = n$.

Sufficiency. For $(shd_1') \implies (hd_1)$ we have

$$h_m^\nu ||A_m^p P_p x|| = ||A_m^n A_n^p P_p x|| \leq N h_n^\nu h_n^\epsilon \frac{h_{n+1}}{h_n} ||A_n^p P_p x||$$

for all $(m, n, p, x) \in T \times X$.

For $(hd_2') \implies (hd_2)$ we have

$$||A_m^p Q_p x|| = ||A_m^n A_n^p Q_p x|| \geq \frac{1}{N} \left(\frac{h_m}{h_n} \right)^\nu h_m^{-\epsilon} \frac{h_m}{h_{m+1}} ||A_n^p Q_p x||$$

for all $(m, n, p) \in T \times X$. □

Remark 4.3. *1. If $\epsilon = 0$, we obtain the concept of uniform h-dichotomy (u.h.d.).*

2. If $h_n = e^n$, for all $n \in \mathbb{N}$ we obtain the concept of exponential dichotomy (e.d.)

3. If $h_n = n + 1$, for all $m \in \mathbb{N}$ we obtain the concept polynomial dichotomy (p.d.)

4. If $h_n = \sqrt{n^2 + 1}$, for all $m \in \mathbb{N}$ we obtain the concept dichotomy with radical.

Definition 4.4. *If then the pair $(\mathcal{A}, P)$ is strongly h-dichotomic (h.d.) then there are $N \geq 1, \epsilon \geq 0$ and $\nu > 0$ such that:*

(shd_1) $h_m^\nu ||A_m^p P_p x|| \leq N h_n^\nu h_n^\epsilon ||A_n^p P_p x||$

(shd_2) $h_m^\nu ||A_n^p Q_p x|| \leq N h_n^\nu h_m^\epsilon ||A_m^p Q_p x||$

for all $(m, n, p, x) \in T \times X$.

Remark 4.5. *The pair (A, P) is strongly h-dichotomic if and only if there are $N \geq 1$, $\nu > 0$ and $\epsilon \geq 0$ such that:*

$(shd_1^{'})$ $h_m^\nu ||A_m^n P_n x|| \leq N h_n^\nu h_n^\epsilon ||P_n x||$

$(shd_2^{'})$ $h_m^\nu ||Q_n x|| \leq N h_n^\nu h_m^\epsilon ||A_m^n Q_n x||$

for all $(m, n, x) \in \Delta \times X$.

Proof. Necessity. Results from Definition 8 for $p = n$.

Sufficiency. For $(shd_1') \implies (shd_1)$ we have

$$h_m^\nu ||A_m^p P_p x|| = h_m^\nu ||A_m^n A_n^p P_p x|| \leq N h_n^\nu h_n^\epsilon ||A_n^p P_p x||$$

for all $(m,n,p,x) \in T \times X$.

For $(shd_2') \implies (hd_2)$ we have

$$||A_m^p Q_p x|| = ||A_m^n A_n^p Q_p x|| \geq \frac{1}{N}\left(\frac{h_m}{h_n}\right)^{\nu} h_m^{-\epsilon}||A_n^p Q_p x||$$

for all $(m,n,p) \in T \times X$. □

Remark 4.6. *1. If $\epsilon = 0$, we obtain the concept of strong uniform h-dichotomy (s.u.h.d.).*

2. If $h_n = e^n$, for all $n \in \mathbb{N}$ we obtain the concept of strong exponential dichotomy (s.e.d.)

3. If $h_n = n+1$, for all $m \in \mathbb{N}$ we obtain the concept strong polynomial dichotomy (s.p.d.)

4. If $h_n = \sqrt{n^2+1}$, for all $m \in \mathbb{N}$ we obtain the concept dichotomy with radical.

Proposition 4.7. *Let (P_n) strong invariant. The pair (A,P) is strongly h-dichotomic if and only if there are $N \geq 1$, $\nu > 0$ and $\epsilon \geq 0$ such that:*

$(hd_1^{s''})$ $h_m^{\nu}||A_m^n P_n x|| \leq N h_n^{\nu} h_n^{\epsilon}||P_n x||$

$(hd_2^{s''})$ $h_m^{\nu}||B_m^n Q_m x|| \leq N h_n^{\nu} h_m^{\epsilon}||Q_m x||$

for all $(m,n,x) \in \Delta \times X$.

Proof. Neccessity. The implication $(hd_1') \implies (hd_1^{s'})$ is immediate for Remark 13. We will to prove $(hd_2') \implies (hd_2^{s'})$. From (hd_2') and Remark 5 we have

$$h_m^{\nu}||Q_n B_n^m Q_m x|| = h_m^{\nu}||Q_n B_n^m Q_m x|| \leq$$

$$\leq N h_n^{\nu} h_m^{\epsilon}||A_m^n Q_n B_n^m Q_m x|| \leq N h_n^{\nu} h_m^{\epsilon}||Q_m A_m^n B_n^m Q_m x|| \leq$$

$$\leq N h_n^{\nu} h_m^{\epsilon}||Q_m x||$$

for all $(m,n,x) \in \Delta \times X$.

Sufficiency. The implication $(hd_1^{s'}) \implies (hd_1')$ is immediate for Remark 13. We will to prove $(hd_2^{s'}) \implies (hd_2')$. From $(hd_2^{s'})$ and Remark 5 we have

$$h_m^{\nu}||Q_n x|| = h_m^{\nu}||B_n^m A_m^n Q_n x|| \leq N h_n^{\nu} h_m^{\epsilon}||A_m^n Q_n x||$$

for all $(m,n,x) \in \Delta \times X$. □

Proposition 4.8. *Let (P_n) strong invariant. The pair (A,P) is h-dichotomic if and only if there are $N \geq 1$, $\nu > 0$ and $\epsilon \geq 0$ such that:*

$(shd_1^{s''})$ $h_m^{\nu}||A_m^n P_n x|| \leq N h_n^{\nu} \dfrac{h_{n+1}}{h_n} h_n^{\epsilon}||P_n x||$

$(shd_2^{s''})$ $h_m^{\nu}||B_n^m Q_m x|| \leq N h_n^{\nu} \dfrac{h_{m+1}}{h_m} h_m^{\epsilon}||Q_m x||$

for all $(m,n,x) \in \Delta \times X$.

Proof. Neccessity. The implication $(shd_1^{'}) \implies (shd_1^{s'})$ is immediate for Remark 11. We will to prove $(shd_2^{'}) \implies (shd_2^{s'})$. From $(shd_2^{'})$ and Remark 5 we have

$$h_m^{\nu}||Q_nB_n^mQ_mx|| = h_m^{\nu}||Q_nB_n^mQ_mx|| \le$$

$$\le Nh_n^{\nu}h_m^{\epsilon}\frac{h_{m+1}}{h_m}||A_m^nQ_nB_n^mQ_mx|| \le Nh_n^{\nu}h_m^{\epsilon}\frac{h_{m+1}}{h_m}||Q_mA_m^nB_n^mQ_mx|| \le$$

$$\le Nh_n^{\nu}h_m^{\epsilon}\frac{h_{m+1}}{h_m}||Q_mx||$$

for all $(m,n,x) \in \Delta \times X$.

Sufficiency. The implication $(shd_1^{s'}) \implies (shd_1^{'})$ is immediate for Remark 11. We will to prove $(shd_2^{s'}) \implies (shd_2^{'})$. From $(shd_2^{s'})$ and Remark 4 we have

$$h_m^{\nu}||Q_nx|| = h_m^{\nu}||B_n^mA_m^nQ_nx|| \le Nh_n^{\nu}h_m^{\epsilon}\frac{h_{m+1}}{h_m}||A_m^nQ_nx||$$

for all $(m,n,x) \in \Delta \times X$. □

5 The main results

In what follows, we will give some characterizations of Datko type for dichotomy with growth rates for linear discrete time systems in Banach spaces with respect to invariant and strongly invariant projections sequences.

In order to do this, we will consider $\mathcal{H}$ the set of growth rates (h_n) that satisfy the following properties:

(1) $\exists H > 1 : h_{n+1} \le Hh_n, \forall n \in \mathbb{N}$

(2) $\forall \alpha \in (-1,0),\ \exists\ H_1 > 1 : \sum\limits_{k=m}^{\infty} h_k^{\alpha} \le H_1h_m^{\alpha},\ \forall m \in \mathbb{N}$

(3) $\forall \alpha \in (0,1),\ \exists\ H_2 > 1 : \sum\limits_{j=0}^{m} h_j^{\alpha} \le H_2h_m^{\alpha},\ \forall m \in \mathbb{N}.$

Theorem 5.1. *The pair $(\mathcal{A}, P)$ is h-dichotomic if and only if there exist $D \ge 1, d > 0$, $\epsilon \ge 0$ with:*

(hD_1) $\sum\limits_{k=n}^{\infty} \frac{h_{k+1}}{h_k}h_k^d||A_k^pP_px|| \le Dh_n^{\epsilon+d}||A_n^pP_px||$

(hD_2) $\sum\limits_{j=n}^{m} \frac{h_{j+1}}{h_j}h_j^{-d}||A_j^pQ_px|| \le Dh_m^{\epsilon-d}||A_m^pQ_px||$

for all $(m,n,p,x) \in T \times X$.

Proof. Necessity: Let be $d \in (0, \nu)$.
(hD_1)

$$\sum_{k=n}^{\infty} \frac{h_{k+1}}{h_k} h_k^d ||A_k^p P_p x|| \leq \sum_{k=n}^{\infty} \frac{h_{k+1}}{h_k} h_k^d N h_n^{\epsilon} \left(\frac{h_k}{h_n}\right)^{-\nu} \frac{h_{n+1}}{h_n} ||A_n^p P_p x|| \leq$$

$$\leq N H_1 h_n^{\epsilon} h_n^{\nu} ||A_n^p P_p x|| \sum_{k=n}^{\infty} \frac{h_{k+1}}{h_k} h_k^{d-\nu} \leq N H_1^2 H_2 h_n^{\epsilon} h_n^{\nu} h_n^{d-\nu} ||A_n^p P_p x|| \leq$$

$$\leq N H_1^2 H_2 h_n^{\epsilon+d} ||A_n^p P_p x|| = D h_n^{\epsilon+d} ||A_n^p P_p x||$$

where $D = N H_1^2 H_2 \geq 1$ and $(m, n, p, x) \in T \times X$.
(hD_2)

$$\sum_{j=n}^{m} \frac{h_{j+1}}{h_j} h_j^{-d} ||A_j^p Q_p x|| \leq \sum_{j=n}^{m} \frac{h_{j+1}}{h_j} h_j^{-d} N h_m^{\epsilon} \left(\frac{h_m}{h_j}\right)^{-\nu} \frac{h_{m+1}}{h_m} ||A_m^p Q_p x|| \leq$$

$$\leq N H_1 h_m^{\epsilon} h_m^{-\nu} ||A_m^p Q_p x|| \sum_{j=n}^{m} \frac{h_{j+1}}{h_j} h_j^{\nu-d} \leq N H_1^2 H_3 h_m^{\epsilon} h_m^{-\nu} h_m^{\nu-d} ||A_m^p Q_p x|| \leq$$

$$\leq N H_1^2 H_3 h_m^{\epsilon-d} ||A_m^p Q_p x|| = D h_m^{\epsilon-d} ||A_m^p Q_m x||$$

where $D = N H_1^2 H_3 \geq 1$ and $(m, n, p, x) \in T \times X$.

Sufficiency For $k = m$ in (hD_1) and $j = n$ in (hD_2), we obtain (hd_1), respectively (hd_2). □

Corollary 5.2. *The pair $(\mathcal{A}, P)$ is uniformly h-dichotomic if and only if there exist $D \geq 1, d > 0$, with:*

(uhD_1) $\displaystyle\sum_{k=n}^{\infty} \frac{h_{k+1}}{h_k} h_k^d ||A_k^p P_p x|| \leq D h_n^d ||A_n^p P_p x||$

(uhD_2) $\displaystyle\sum_{j=n}^{m} \frac{h_{j+1}}{h_j} h_j^{-d} ||A_j^p Q_p x|| \leq D h_m^{-d} ||A_m^p Q_p x||$

for all $(m, n, p, x) \in T \times X$.

Proof. Results immediate from Theorem 5 for $\epsilon = 0$. □

Corollary 5.3. *Let $h \in \mathcal{H}$. The pair $(\mathcal{A}, P)$ is exponentially dichotomic if and only if there are $D \geq 1, d > 0$, and $\epsilon \geq 0$ with:*

(eD_1) $\displaystyle\sum_{k=n}^{\infty} e^{dk} ||A_k^p P_p x|| \leq D e^{n(\epsilon+d)} ||A_n^p P_p x||$

(eD_2) $\displaystyle\sum_{j=n}^{m} e^{-dj} ||A_j^p Q_p x|| \leq D e^{m(\epsilon-d)} ||A_m^p Q_p x||$

for all $(m, n, p, x) \in T \times X$.

Proof. It follows from Theorem 5 for $h_m = e^m$. □

Remark 5.4. *The preceding theorems are variants for the case of polynomial dichotomy property of well-known theorems due to Popa et al. [33,34] for exponential stability and exponential dichotomy.*

Theorem 5.5. *The pair* $(\mathcal{A}, P)$ *is strongly h-dichotomic if and only if there exist* $D \geq 1, d > 0,\ \epsilon \geq 0$ *with:*

(shD_1) $\sum_{k=n}^{\infty} h_k^d ||A_k^p P_p x|| \leq D h_n^{\epsilon+d} ||A_n^p P_p x||$

(shD_2) $\sum_{j=n}^{m} h_j^{-d} ||A_j^p Q_p x|| \leq D h_m^{\epsilon-d} ||A_m^p Q_p x||$

for all $(m, n, p, x) \in T \times X$.

Proof. Necessity. Let be $d \in (0, \nu)$.
(shD_1)

$$\sum_{k=n}^{\infty} h_k^d ||A_k^p P_p x|| \leq \sum_{k=n}^{\infty} h_k^d N h_n^{\epsilon} \left(\frac{h_k}{h_n}\right)^{-\nu} ||A_n^p P_p x|| \leq$$

$$\leq N h_n^{\epsilon} h_n^{\nu} ||A_n^p P_p x|| \sum_{k=n}^{\infty} h_k^{d-\nu} \leq N H_2 h_n^{\epsilon} h_n^{\nu} h_n^{d-\nu} ||A_n^p P_p x|| \leq$$

$$\leq N H_2 h_n^{\epsilon+d} ||A_n^p P_p x|| = D h_n^{\epsilon+d} ||A_n^p P_p x||$$

where $D = N H_2 \geq 1$ and $(m, n, p, x) \in T \times X$.
(shD_2)

$$\sum_{j=n}^{m} h_j^{-d} ||A_j^p Q_p x|| \leq \sum_{j=n}^{m} h_j^{-d} N h_m^{\epsilon} \left(\frac{h_m}{h_j}\right)^{-\nu} ||A_m^p Q_p x|| \leq$$

$$\leq N h_m^{\epsilon} h_m^{-\nu} ||A_m^p Q_p x|| \sum_{j=n}^{m} h_j^{\nu-d} \leq N H_3 h_m^{\epsilon} h_m^{-\nu} h_m^{\nu-d} ||A_m^p Q_p x|| \leq$$

$$\leq N H_3 h_m^{\epsilon-d} ||A_m^p Q_p x|| = D h_m^{\epsilon-d} ||A_m^p Q_m x||$$

where $D = N H_3 \geq 1$ and $(m, n, p, x) \in T \times X$.

Sufficiency. For $k = m$ in (shD_1) and $j = n$ in (shD_2) we obtain (shd_1) and (shd_2). □

Corollary 5.6. *The pair* $(\mathcal{A}, P)$ *is uniformly strongly h-dichotomic if and only if there exist* $D \geq 1, d > 0$, *with:*

$(ushD_1)$ $\sum_{k=n}^{\infty} h_k^d ||A_k^p P_p x|| \leq D h_n^d ||A_n^p P_p x||$

$$(ushD_2)\ \sum_{j=n}^{m} h_j^{-d}||A_j^p Q_p x|| \leq D h_m^{-d}||A_m^p Q_p x||$$

for all $(m,n,p,x) \in T \times X$.

Proof. Results immediate from Theorem 12 for $\epsilon = 0$. □

Corollary 5.7. *Let* $h \in \mathcal{H}$. *The pair* $(\mathcal{A}, P)$ *is strongly exponentially dichotomic if and only if there are* $D \geq 1, d > 0$, *and* $\epsilon \geq 0$ *with:*

$$(seD_1)\ \sum_{k=n}^{\infty} e^{dk}||A_k^p P_p x|| \leq D e^{n(\epsilon+d)}||A_n^p P_p x||$$

$$(seD_2)\ \sum_{j=n}^{m} e^{-dj}||A_j^p Q_p x|| \leq D e^{m(\epsilon-d)}||A_m^p Q_p x||$$

for all $(m,n,p,x) \in T \times X$.

Proof. It follows from Theorem 12 for $h_m = e^m$. □

In what follows, we will extend the characterizations obtained above, for the case of h- dichotomy and strong h-dichotomy with growth rates for discrete systems with respect to strongly invariant projections sequences. Therefore, next we will consider the linear system $(\mathcal{A})$ and (P_n) a projections sequences which is strongly invariant to $(\mathcal{A})$.

Theorem 5.8. *If the pair* $(\mathcal{A}, P)$ *is h-dichotomic then there exist* $D \geq 1, d > 0$, $\epsilon \geq 0$ *with:*

$$(hD_1^{s'})\ \sum_{k=n}^{\infty} \frac{h_{k+1}}{h_k} h_k^d ||A_k^p P_p x|| \leq D h_n^{\epsilon+d}||A_n^p P_p x||$$

$$(hD_2^{s'})\ \sum_{j=n}^{m} \frac{h_{j+1}}{h_j} h_j^{-d} ||B_j^m Q_m x|| \leq D h_m^{\epsilon-d}||Q_m x||$$

for all $(m,n,p,x) \in T \times X$.

Proof. Necessity. Let be $d \in (0, \nu)$.
The implication $(hd_1^{s'}) \implies (hD_1^{s'})$ is immediate from Theorem 12.
$(hd_2^{s'}) \implies (hD_2^{s'})$

$$\sum_{j=n}^{m} \frac{h_{j+1}}{h_j} \cdot h_j^{-d}||B_j^m Q_m x|| = \sum_{j=n}^{m} \frac{h_{j+1}}{h_j} \cdot h_j^{-d}||Q_j B_j^m Q_m x|| \leq$$

$$\leq \sum_{j=n}^{m} \frac{h_{j+1}}{h_j} \cdot h_j^{-d} N h_n^{\epsilon} \cdot \frac{h_{m+1}}{h_m} \left(\frac{h_j}{h_m}\right)^{\nu} ||A_m^j Q_j B_n^m Q_m x|| =$$

$$= NH_1 h_m^{-\nu+\epsilon}||Q_m x|| \sum_{j=n}^{m} \frac{h_{j+1}}{h_j} h_j^{-d+\nu} \leq$$

$$\leq NH_1^2 h_m^{-\nu+\epsilon}||Q_m x|| \sum_{j=n}^{m} h_j^{-d+\nu} \leq$$

$$\leq NH_1^2 H_3 h_m^{-\nu+\epsilon} h_m^{-d+\nu}||Q_m x|| = Dh_m^{\epsilon-d}||Q_m x|| \leq Dh_m^{\epsilon-d}||Q_m x||$$

where $D = NH_1^2 H_3 \geq 1$ and $(m, n, p, x) \in T \times X$.

Sufficiency. For $j = n$ in $(hD_2^{s'})$. $\square$

Corollary 5.9. *The pair $(\mathcal{A}, P)$ is uniformly h-dichotomic if and only if there exist $D \geq 1, d > 0$, with:*

$(uhD_1^{s'})$ $\sum_{k=n}^{\infty} \frac{h_{k+1}}{h_k} h_k^d ||A_k^p P_p x|| \leq Dh_n^d ||A_n^p P_p x||$

$(uhD_2^{s'})$ $\sum_{j=n}^{m} \frac{h_{j+1}}{h_j} h_j^{-d} ||B_m^j Q_m x|| \leq Dh_m^{-d}||Q_m x||$

for all $(m, n, p, x) \in T \times X$.

Proof. Results immediat from Theorem 13 for $\epsilon = 0$. $\square$

Corollary 5.10. *Let $h \in \mathcal{H}$. The pair $(\mathcal{A}, P)$ is exponentially dichotomic if only if there are $D \geq 1, d > 0$, and $\epsilon \geq 0$ with:*

$(eD_1^{s'})$ $\sum_{k=n}^{\infty} e^{dk} ||A_k^p P_p x|| \leq e^{n(\epsilon+d)}||A_n^p P_p x||$

$(eD_2^{s'})$ $\sum_{j=n}^{m} e^{-dj} ||B_m^j Q_m x|| \leq De^{m(\epsilon-d)}||Q_m x||$

for all $(m, n, p, x) \in T \times X$.

Proof. It follows from Theorem 13 for $h_m = e^m$. $\square$

Theorem 5.11. *The pair $(\mathcal{A}, P)$ is strongly h-dichotomic if and only if there exist $D \geq 1, d > 0$, $\epsilon \geq 0$ with:*

$(shD_1^{s'})$ $\sum_{k=n}^{\infty} h_k^d ||A_k^p P_p x|| \leq Dh_n^{\epsilon+d}||A_n^p P_p x||$

$(shD_2^{s'})$ $\sum_{j=n}^{m} h_j^{-d} ||B_m^j Q_m x|| \leq Dh_m^{\epsilon-d}||Q_m x||$

for all $(m, n, p, x) \in T \times X$.

Proof. *Necessity.* Let be $d \in (0, \nu)$.
The implication $(shd_1^{s'}) \implies (shD_1^{s'})$ is immediate from Theorem 5.
$(shd_2^{s'}) \implies (shD_2^{s'})$

$$\sum_{j=n}^{m} h_j^{-d} ||B_j^m Q_m x|| = \sum_{j=n}^{m} h_j^{-d} ||Q_j B_j^m Q_m x|| \leq$$

$$\leq \sum_{j=n}^{m} h_j^{-d} N h_n^{\epsilon} \left(\frac{h_j}{h_m}\right)^{\nu} ||A_m^j Q_j B_n^m Q_m x|| =$$

$$= N h_m^{-\nu+\epsilon} ||Q_m x|| \sum_{j=n}^{m} h_j^{-d+\nu} \leq$$

$$\leq N h_m^{-\nu+\epsilon} ||Q_m x|| \sum_{j=n}^{m} h_j^{-d+\nu} \leq$$

$$\leq N H_3 h_m^{-\nu+\epsilon} h_m^{-d+\nu} ||Q_m x|| \leq D h_m^{\epsilon-d} ||Q_m x||$$

where $D = NH_3 \geq 1$ and $(m, n, p, x) \in T \times X$.

Sufficiency. Results immediate from $(hD_2^{s'})$ for $j = n$.

□

Corollary 5.12. *If the pair $(\mathcal{A}, P)$ is uniformly strongly h-dichotomic if and only if there exist $D \geq 1, d > 0$, with:*

$(ushD_1^{s'})$ $\sum_{k=n}^{\infty} h_k^d ||A_k^p P_p x|| \leq D h_n^d ||A_n^p P_p x||$

$(ushD_2^{s'})$ $\sum_{j=n}^{m} h_j^{-d} ||B_m^j Q_m x|| \leq D h_m^{-d} ||Q_m x||$

for all $(m, n, p, x) \in T \times X$.

Proof. Results immediat from Theorem 14 for $\epsilon = 0$. □

Corollary 5.13. *Let $h \in \mathcal{H}$. The pair $(\mathcal{A}, P)$ is strongly exponentially dichotomic if and only if there are $D \geq 1, d > 0$, and $\epsilon \geq 0$ with:*

$(seD_1^{s'})$ $\sum_{k=n}^{\infty} e^{dk} ||A_k^p P_p x|| \leq e^{n(\epsilon+d)} ||A_n^p P_p x||$

$(seD_2^{s'})$ $\sum_{j=n}^{m} e^{-dj} ||B_m^j Q_m x|| \leq D e^{m(\epsilon-d)} ||Q_m x||$

for all $(m, n, p, x) \in T \times X$.

Proof. It follows from Theorem 14 for $h_m = e^m$. □

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In memory of Professor Borislav D. CRSTICI on the Centenary of his birth.

CLASSIFYING SPACES FOR FAMILIES OF SUBGROUPS FOR 8 -LOCATED GROUPS

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Abstract

We construct a low-dimensional classifying space for the family of virtually cyclic subgroups of a group acting properly on an 8-located complex with the SD'-property. The key property we use is that the minimal displacement set in an 8-located complex with the SD' property embeds isometrically into the complex and it is systolic. [1]

1 Introduction

Curvature can be expressed both in metric and combinatorial terms. On the metric side, one can refer to nonpositively curved in the sense of Aleksandrov and Gromov, i.e. by comparing small triangles in the space with triangles in the Euclidean plane. Such triangles must satisfy the CAT(0) inequality. On the combinatorial side, one can express curvature using a condition, called local 6-largeness which was introduced independently by Chepoi [3] (under the name of bridged complexes), Januszkiewicz-Świątkowski [11] and Haglund [9]. In [1], [4], [5], [6], [18], [12], [13] other conditions of this type are studied.

In [12] we study of a version of 8-location, suggested in [18, Subsection 5.1]. This 8-location says that homotopically trivial loops of length at most 8 admit filling diagrams with one internal vertex. However, in the new 8-location essential

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4-loops are allowed. In [12] (Theorem 4.3) it is shown that this local combinatorial condition is a negative-curvature-type condition. Namely, it is shown that simply connected, 8-located simplicial complexes are Gromov hyperbolic. In [10] we study 7-located complexes as defined initially in [18].

The purpose of the current paper is to construct a low-dimensional classifying space for the family of virtually cyclic subgroups of a group acting properly on an 8-located complex with the SD'-property. This construction relies on the structure of the minimal displacement set in an 8-located complex with the SD'-property. This set was studied for CAT(0) spaces (see [2]), for systolic complexes (see [7], [19]), for weakly systolic complexes (see [15]) and for 8-located complexes with the SD'-property (see [14]). Namely, for systolic, weakly systolic and 8-located complexes with the SD'-property, such set embeds isometrically into the complex and it is systolic. Therefore one can apply certain results proven in [7] and [19] on systolic complexes. As immediate consequence of these results, it follows that any isometry of an 8-located complex with the SD'-property either has a fixed simplex (elliptic case) or it stabilizes a thick geodesic (hyperbolic case). For systolic complexes, studying the structure of the minimal displacement set turns out to be useful when constructing a low-dimensional classifying space for the family of virtually cyclic subgroups of a group acting properly on a systolic complex (see [19]). We obtain similar results for 8-located simplicial complexes with the SD'-property.

2 Preliminaries

Let X be a simplicial complex. We denote by $X^{(k)}$ the k-skeleton of $X, 0 \leq k < \dim X$. A subcomplex L in X is called *full* as a subcomplex of X if any simplex of X spanned by a set of vertices in L, is a simplex of L. For a set $A = \{v_1, ..., v_k\}$ of vertices of X, by $\langle A \rangle$ or by $\langle v_1, ..., v_k \rangle$ we denote the *span* of A, i.e. the smallest full subcomplex of X that contains A. We write $v \sim v'$ if $\langle v, v' \rangle \in X$ (it can happen that $v = v'$). We write $v \nsim v'$ if $\langle v, v' \rangle \notin X$. We call X *flag* if any finite set of vertices which are pairwise connected by edges of X, spans a simplex of X.

A *cycle* (*loop*) γ in X is a subcomplex of X isomorphic to a triangulation of S^1. A *full cycle* in X is a cycle that is full as a subcomplex of X. A k-*wheel* in X $(v_0; v_1, ..., v_k)$ (where $v_i, i \in \{0, ..., k\}$ are vertices of X) is a subcomplex of X such that $\gamma = (v_1, ..., v_k)$ is a full cycle and $v_0 \sim v_1, ..., v_k$. The *length* of γ (denoted by $|\gamma|$) is the number of edges of γ. If $g = (v_1, ..., v_k)$ is a 1-skeleton geodesic of X, the *length* of g (denoted by $|g|$ or by $|(v_1, ..., v_k)|$) is the number of edges of g.

We define the *combinatorial metric* on the 0-skeleton of X as the number of edges in the shortest 1-skeleton path joining two given vertices.

A *ball (sphere)* $B_i(v, X)$ $(S_i(v, X))$ of radius i around some vertex v is a full subcomplex of X spanned by vertices at combinatorial distance at most i (at combinatorial distance i) from v.

Definition 2.1. *A simplicial complex is m-*located *if it is flag and every full homotopically trivial loop of length at most m is contained in a 1-ball.*

Let σ be a simplex of a simplicial complex X. The *link* of X at σ, denoted X_σ, is the subcomplex of X consisting of all simplices of X which are disjoint from σ and which, together with σ, span a simplex of X. We call a flag simplicial complex X, *k-large* if there are no full j-cycles in X, for $j < k$. We say X is *locally k-large* if all its links are k-large. We call X k-large, $k \geq 4$ if it is connected, simply connected and locally k-large.

We introduce further a global combinatorial condition on a flag simplicial complex.

Definition 2.2. *Let X be a flag simplicial complex. For a vertex O of X and a natural number n, we say that X satisfies* the property $SD'_n(O)$ *if for every $i \in \{1, ..., n\}$ we have:*

1. *(T) (triangle condition): for every edge $e \in S_{i+1}(O)$, the intersection $X_e \cap B_i(O)$ is non-empty;*
2. *(V) (vertex condition): for every vertex $v \in S_{i+1}(O)$, and for every two vertices $u, w \in X_v \cap B_i(O)$, there exists a vertex $t \in X_v \cap B_i(O)$ such that $t \sim u, w$.*

We say X satisfies the property $SD'(O)$ *if $SD'_n(O)$ holds for each natural number n. We say X satisfies* the property SD' *if $SD'_n(O)$ holds for each natural number n and for each vertex O of X.*

2.1 Minimal displacement set for systolic complexes

For systolic complexes the minimal displacement set is studied in [7].

Let h be an isometry of a simplicial complex X. We define the *displacement function* $d_h : X^{(0)} \to \mathbf{N}$ by $d_h(x) = d_X(h(x), x)$. The *translation length* of h is defined as $|h| = \min_{x \in X^{(0)}} d_h(x)$. If h does not fix any simplex of X, then h is called *hyperbolic*. In such case one has $|h| > 0$. Otherwise we call the isometry h *elliptic*. For a hyperbolic isometry h, we define the minimal displacement set $\mathrm{Min_X(h)}$ as the subcomplex of X spanned by the set of vertices where d_h attains its minimum. Clearly, $\mathrm{Min_X(h)}$ is invariant under the action of h.

Theorem 2.3. *Let h be a hyperbolic isometry of a systolic complex X. Then the subcomplex* $\mathrm{Min_X(h)}$ *is a systolic complex, isometrically embedded into X (see [7], Propositions* 3.3 *and* 3.4*).*

Let h be an isometry of a simplicial complex X. An h-invariant geodesic in X is called an *axis* of h. We say that $\mathrm{Min_X(h)}$ is the union of axes, if for every vertex $x \in \mathrm{Min_X(h)}$, there is an h-invariant geodesic passing through x, i.e. $\mathrm{Min_X(h)}$ can be written as follows:

$$\mathrm{Min_X(h)} = \mathrm{span}\{\bigcup \gamma | \gamma \text{ is an } h\text{-invariant geodesic } \} \quad (2.1)$$

In this case, the isometry h acts on X as a translation along the axes by the number $|h|$.

For two subcomplexes $X_1, X_2 \subset X$, the distance $d_{\min}(X_1, X_2)$ is defined to be

$$d_{\min}(X_1, X_2) = \min\{d_X(x_1, x_2) | x_1 \in X_1, x_2 \in X_2\}.$$

We are ready now to define the *graph of axes*. For a hyperbolic isometry h satisfying (2.1), we define the simplicial graph Y_h as follows:

$$Y_h^{(0)} = \{\gamma | \gamma \text{ is an } h\text{-invariant geodesic in } \mathrm{Min_X(h)}\},$$
$$Y_h^{(1)} = \{\{\gamma_1, \gamma_2\} | d_{\min}(\gamma_1, \gamma_2) \leq 1\}.$$

Let $d_{Y(h)}$ denote the associated metric on $Y_h^{(0)}$.

2.2 Classifying spaces with finite or virtually cyclic stabilisers

The main goal of this section is, given a group G, to describe a method of constructing a model for a classifying space with virtually cyclic stabilisers out of a model for a classifying space with finite stabilisers. The presented method is due to W. Lück and M. Weiermann ([17]). First we give the necessary definitions.

A collection of subgroups $\mathcal{F}$ of a group G is called a *family* if it is closed under taking subgroups and conjugation by elements of G. Two examples which will be of interest to us are the family $\mathcal{FIN}$ of all finite subgroups, the family $\mathcal{VCY}$ of all virtually cyclic subgroups.

Definition 2.4. *Given a group G and a family of its subgroups $\mathcal{F}$, a* model for the classifying space $E_{\mathcal{F}}G$ *is a G-CW-complex X such that for any subgroup $H \subset G$ the fixed point set X^H is contractible if $H \in \mathcal{F}$, and empty otherwise.*

Let $\underline{E}G$ denote $E_{\mathcal{FIN}}G$ and let $\underline{\underline{E}}G$ denote $E_{\mathcal{VCY}}G$.

A model for $E_{\mathcal{F}}G$ exists for any group and any family. Any two models for $E_{\mathcal{F}}G$ are G-homotopy equivalent (see [16]). However, general constructions always produce infinite dimensional models.

We will describe a method of constructing a finite dimensional model for $\underline{\underline{E}}G$ out of a model for $\underline{E}G$ and appropriate models associated to infinite virtually cyclic subgroups of G. If $H \subset G$ is a subgroup and $\mathcal{F}$ is a family of subgroups of G, let $\mathcal{F} \cap H$ denote the family of all subgroups of H which belong to the family

$\mathcal{F}$. More generally, if $\phi : H \to G$ is a homomorphism, let $\phi^\star \mathcal{F}$ denote the smallest family of subgroups of H that contains $\phi^{-1}(F)$ for all $F \in \mathcal{F}$.

Consider the collection $\mathcal{VCY} \setminus \mathcal{FIN}$ of infinite virtually cyclic subgroups of G. It is not a family since it does not contain the trivial subgroup. Define an equivalence relation on $\mathcal{VCY} \setminus \mathcal{FIN}$ by

$$H_1 \sim H_2 \iff |H_1 \cap H_2| = \infty$$

Let $[H]$ denote the equivalence class of H, and let $[\mathcal{VCY} \subset \mathcal{FIN}]$ denote the set of equivalence classes. The group G acts on $[\mathcal{VCY} \subset \mathcal{FIN}]$ by conjugation, and for a class $[H] \in [\mathcal{VCY} \subset \mathcal{FIN}]$ define the subgroup $N_G(H) \subseteq G$ to be the stabiliser of $[H]$ under this action, i.e.

$$N_G(H) = \{g \in G | \ |g^{-1}Hg \cap H| = \infty\}$$

The subgroup $N_G(H)$ is called the *commensurator* of H, since its elements conjugate H to the subgroup commensurable with H. For $[H] \in [\mathcal{VCY} \subset \mathcal{FIN}]$ define the family $\mathcal{G}[H]$ of subgroups of $N_G(H)$ as follows

$$\mathcal{G}[H] = \{K \subset G | \ K \in [\mathcal{VCY} \subset \mathcal{FIN}], [K] = [H]\} \cup \{K \in \mathcal{FIN} \cap N_G[H]\}.$$

Definition 2.5. *A group G satisfies* condition (C) *if for every $g, h \in G$ with $|h| = \infty$ (infinite order) and for any $k, l \in \mathbf{Z}$, we have*

$$gh^k g^{-1} = h^l \implies |k| = |l|$$

Lemma 2.6. *Let $K \subset N_G[H]$ be a finitely generated subgroup that contains some representative of $[H]$ and assume that the group G satisfies condition (C). Choose an element $h \in H$ such that $[\langle h \rangle] = [H]$ (any element of infinite order has this property). Then there exists an integer $k \geq 1$, such that $\langle h^k \rangle$ is normal in K.*

For the proof see [19], Lemma 2.6, page 8.

3 Minimal displacement set for 8 - located simplicial complexes with the SD'-property

The minimal displacement set for 8-located complexes with the SD'-property is studied in [14]. The main result obtained there is given below.

Theorem 3.1. *For a (simplicial) isometry h with no fixed simplices of an 8 - located simplicial complex X with the SD'-property, the 1-skeleton of* $\mathrm{Min_X(h)}$ *is isometrically embedded into X.*

For the proof see [14], Theorem 3.3.

Theorem 3.2. *Let X be an 8-located simplicial complex with the SD'-property. Then the 1-skeleton of X, equipped with the standard path metric, is Gromov hyperbolic.*

For the proof see [14], Corollary 3.4.

Theorem 3.3. *Let h be a (simplicial) isometry with no fixed simplices of an 8-located complex X with the SD'-property. Then the set* $\mathrm{Min_X(h)}$ *is systolic.*

Proof. Due to Theorem 3.1, the 1-skeleton of $\mathrm{Min_X(h)}$ is isometrically embedded into X. Moreover, Theorem 3.2 implies that $\mathrm{Min_X(h)}$ is Gromov hyperbolic. Then $\mathrm{Min_X(h)}$ is 7-systolic and therefore systolic. □

The following results on 8-located complexes with the SD'-property are immediate consequences of the fact that the minimal displacement set of a nonelliptic isometry acting on such complex is systolic and it embeds isometrically into the complex. Their systolic analogues, also given below, imply these similarities.

Theorem 3.4. *Let h be a hyperbolic simplicial isometry of a uniformly locally finite systolic complex X. Then there is an h^n-invariant geodesic for some $n \geq 1$.*

For the proof see [7], Theorem 3.5, page 46.

Theorem 3.5. *Let h be a hyperbolic simplicial isometry of a uniformly locally finite 8-located complex X with the SD'-property. Then there is an h^n-invariant geodesic for some $n \geq 1$.*

Proof. Theorem 3.3 implies that $\mathrm{Min_X(h)}$ is systolic. Then, by Theorem 3.4, in $\mathrm{Min_X(h)}$ there is an h^n-invariant geodesic for some $n \geq 1$. By Theorem 3.1, $\mathrm{Min_X(h)}^{(1)}$ is isometrically embedded into X. Then this h^n-invariant geodesic also belongs to X. This completes the proof.

□

Theorem 3.6. *Let h be a simplicial isometry of a uniformly locally finite systolic complex X. Then either there is an h-invariant simplex (elliptic case) or there is an h-invariant thick geodesic (hyperbolic case).*

For the proof see [7], Theorem 3.8, page 49.

Theorem 3.7. *Let h be a simplicial isometry of a uniformly locally finite 8-located complex X with the SD'-property. Then either there is an h-invariant simplex (elliptic case) or there is an h-invariant thick geodesic (hyperbolic case).*

Proof. Let $Y = \mathrm{Min_X(h)}$. Theorem 3.3 implies that Y is systolic. Then, by Theorem 3.6, in Y either there is an h-invariant simplex (elliptic case) or there is an h-invariant thick geodesic (hyperbolic case). Since, by Theorem 3.1, $Y^{(1)}$ is isometrically embedded into X, either this h-invariant simplex or this h-invariant thick geodesic, also belongs to X.

□

Theorem 3.8. *Let h be a hyperbolic simplicial isometry of a uniformly locally finite systolic complex X. If there exists an h^n-invariant geodesic in X, then for any vertex $x \in \mathrm{Min_X(h^n)} \subset \mathrm{X}$, there exists an h^n-invariant geodesic passing through x.*

For the proof see [7], Remark page 48.

Theorem 3.9. *Let h be a hyperbolic simplicial isometry of a uniformly locally finite 8-located complex X with the SD'-property. If there exists an h^n-invariant geodesic in X, then for any vertex $x \in \mathrm{Min_X(h^n)} \subset \mathrm{X}$, there exists an h^n-invariant geodesic passing through x.*

Proof. Let $Y = \mathrm{Min_X(h)}$. Theorem 3.3 implies that Y is systolic. According to Theorem 3.4, in Y (and then, by Theorem 3.5, also in X) there exists an h^n-invariant geodesic for some $n \geq 1$. Hence, by Theorem 3.8, for any vertex $x \in \mathrm{Min_X(h^n)} \subset \mathrm{Y}$, there exists an h^n-invariant geodesic passing through x. By Theorem 3.1, $Y^{(1)}$ is isometrically embedded into X. Then for any vertex $x \in \mathrm{Min_X(h^n)} \subset \mathrm{X}$, there exists an h^n-invariant geodesic passing through x. □

4 Classifying spaces with virtually cyclic stabiliser for 8-located groups with the SD'-property

In this section we construct a low-dimensional classifying space for the family of virtually cyclic subgroups of a group acting properly on an 8-located complex with the SD'-property.

Theorem 4.1. *Let h be a hyperbolic isometry of a uniformly locally finite systolic complex X such that the subcomplex $\mathrm{Min_X(h)}$ is the union of axes. Then there is a quasi-isometry $c : (Y(h) \times \mathbf{Z}, d_h) \to (\mathrm{Min_X(h)}, \mathrm{d_X})$ where the metric d_h is defined as $d_h((\gamma_1, t_1), (\gamma_2, t_2)) = d_{Y(h)}(\gamma_1, \gamma_2) + |t_1 - t_2|$, and d_X is the metric induced from X.*

For the proof see [19], Theorem 3.3, page 12.

Theorem 4.2. *Let h be a hyperbolic isometry of a uniformly locally finite 8-located complex X with the SD'-property such that the subcomplex $\mathrm{Min_X(h)}$ is the union of axes. Then there is a quasi-isometry $c : (Y(h) \times \mathbf{Z}, d_h) \to (\mathrm{Min_X(h)}, \mathrm{d_X})$ where the metric d_h is defined as $d_h((\gamma_1, t_1), (\gamma_2, t_2)) = d_{Y(h)}(\gamma_1, \gamma_2) + |t_1 - t_2|$, and d_X is the metric induced from X.*

Proof. Due to Theorem 3.1, the 1-skeleton of $\mathrm{Min_X(h)}$ is isometrically embedded into X. Moreover, Theorem 3.3 implies that $\mathrm{Min_X(h)}$ is systolic. Hence the result follows by Theorem 4.1. □

Theorem 4.3. *Let X be a locally finite systolic simplicial complex. For a hyperbolic isometry h whose minimal displacement set is a union of axes (that is, for h satisfying* (2.1)*), the graph of axes $(Y(h), d_{Y(h)})$ is quasi-isometric to a simplicial tree.*

For the proof see [19], Corollary 4.6, page 21.

Theorem 4.4. *Let X be a locally finite* 8*-located complexes with the SD'-property. For a hyperbolic isometry h whose minimal displacement set is a union of axes (that is, for h satisfying* (2.1)*), the graph of axes $(Y(h), d_{Y(h)})$ is quasi-isometric to a simplicial tree.*

Proof. If there do not exist h-invariant geodesics in X, take an h^n-invariant geodesic in X, $n > 1$. Assume there exist h-invariant geodesics in X. These geodesics belong to $\mathrm{Min_X(h)}$ because, according to (2.1),

$$\mathrm{Min_X(h)} = \mathrm{span}\{\bigcup \gamma | \gamma \text{ is an } h\text{-invariant geodesic }\}.$$

Due to Theorem 3.1, the 1-skeleton of $\mathrm{Min_X(h)}$ is isometrically embedded into X. Moreover, Theorem 3.3 implies that $\mathrm{Min_X(h)}$ is systolic. Then, by Theorem 4.3, the desired result holds.

□

For the rest of the section, let G be a group acting properly on a uniformly locally finite 8-located complex X with the SD'-property of dimension d. Let h be a hyperbolic isometry of X.

Theorem 4.5. *The systolic complex X is a model for $\underline{E}G$.*

For the proof see [6], Theorem E.

In order to construct models for the commensurators $N_G[H]$, first we show that the group G satisfies Condition (C). Using this, in every finitely generated subgroup $K \subseteq N_G[H]$ that contains H we find a suitable normal cyclic subgroup. As shown in [19] for systolic complexes, the quotient group acts properly on a quasi-tree.

Lemma 4.6. *The group G satisfies condition (C).*

Proof. The proof is similar to the one given in [19] (Lemma 5.2). Take arbitrary $g, h \in G$ such that $|h| = \infty$, and assume there are $k, l \in \mathbf{Z}$ such that $g^{-1}h^k g = h^l$. We have to show that $|k| = |l|$. Since the action of G on X is proper, the element h acts as a hyperbolic isometry. By Theorem 3.5, in X there is an h^n-invariant geodesic for some $n \geq 1$. We get the claim by considering the following sequence of equalities for the translation length:

$$|k| \cdot |h^n| = |h^{k \cdot n}| = |g^{-1} \cdot h^{n \cdot k} \cdot g| = |h^{\pm l \cdot n}| = |l| \cdot |h^n|.$$

The first and the last of the equalities follow from the fact that the translation length of an element can be measured on an invariant geodesic, the second one is an easy calculation and the third one is straightforward. □

Lemma 4.7. *Let K be a finitely generated subgroup of G, and let $h \in K$ a hyperbolic isometry satisfying* (2.1) *such that $\langle h \rangle$ is normal in K. Then the proper action of G on X induces a proper action of $K/\langle h \rangle$ on the graph of axes $Y(h)$.*

Proof. The proof is similar to the one given in [19], Lemma 5.3 in the systolic case. □

Lemma 4.8. *Let h be a hyperbolic isometry of an* 8*-located complex X with the SD'-property. If h satisfies* (2.1)*, then so does h^n for any $n \in \mathbf{Z} \setminus \{0\}$.*

Proof. The result follows due to the fact that an h-invariant geodesic is h^n-invariant. □

Lemma 4.9. *Let K be a finitely generated subgroup of $N_G[H]$ that contains H. Then there is a short exact sequence*

$$0 \to \langle h \rangle \to K \to K/\langle h \rangle \to 0,$$

such that $h \in H$ is of infinite order and the group $K/\langle h \rangle$ is virtually free.

Proof. The proof is similar to the one given in [19], Lemma 5.4. Choose an element of infinite order $\tilde{h} \in H$ satisfying the following two conditions:

1. the set $\mathrm{Min}(\tilde{\mathrm{h}})$ is the union of axes (see (2.1));

2. the translation length $|\tilde{h}| > 3$.

Both conditions above can be ensured by rising $\tilde{h}$ to a sufficiently large power. Indeed, by Lemma 3.5, there exists $n \geq 1$ such that $\tilde{h}^n$ satisfies the first condition above. If $|\tilde{h}^n| \leq 3$, then replace it by $\tilde{h}^{4n}$. The element $\tilde{h}^{4n}$ satisfies both conditions. If an element satisfies the conditions above then, by Lemma 4.8, so does any of its powers. Since G satisfies Condition (C), by Lemma 2.6, there exists an integer $k \geq 1$ such that $\langle \tilde{h}^k \rangle$ is normal in K.

Put $h = \tilde{h}^k$. By Lemma 4.7, the group $K/\langle h \rangle$ acts properly by isometries on the graph of axes $(Y(h), d_{Y(h)})$, which is, by Theorem 4.4, a quasi-tree. In conclusion the group $K/\langle h \rangle$ is virtually free.

□

The proofs of the next results are similar to the one given for systolic complexes in [19] (see Lemma 5.6, Theorem C).

Theorem 4.10. *For every* $[H] \in \mathcal{VCY} \setminus \mathcal{FIN}$ *there exist:*

1. *a* 2*-dimensional model for* $E_{\mathcal{G}[H]} N_G[H]$*;*
2. *a* 3*-dimensional model for* $\underline{E} N_G[H]$*.*

Theorem 4.11. *There exists a model for* $\underline{\underline{E}}G$ *of dimension*

$$\dim \underline{\underline{E}}G = \begin{cases} d{+}1, & \text{if } d \leq 3, \\ d, & \text{if } d \geq 4. \end{cases}$$

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In memory of Professor Borislav D. CRSTICI on the Centenary of his birth.

ON THE STABILITY OF MULTIPLIERS ON BANACH ALGEBRAS

Laura MANOLESCU

Abstract

In this paper, we give new results concerning the Hyers-Ulam stability of multipliers on Banach algebras. So, we extend some results obtained in 2004 by T. Miura, G. Hirasawa and S.E. Takahasi.
[1]

1 Introduction

In 1940, S.M. Ulam (see [21]) raised the following problem:
given an approximately homomorphism, can one find a homomorphism near it?

In 1941, D.H. Hyers [7] gave an affirmative answer to the question of Ulam for addditive Cauchy equation in Banach spaces. For this reason, one says that the Cauchy equation is stable in the sense of Hyers-Ulam. Afterwards, different generalizations of that initial answer of Hyers were obtained. See, for example, the books [3], [8], [10] and the references therein. See also the recent papers of the author [2], [15] and [16].

The stability in the Hyers-Ulam sense of multipliers on the Banach algebra was considered for the first time by T. Miura, G. Hirasawa and S.E. Takahasi in 2004 (see [17]). See also the papers [1], [5], [12] and [18]. In the following, we will extend their results, by giving new results concerning the generalized stability in the sense of Hyers-Ulam of multipliers on Banach algebras.

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We denote by A a Banach algebra and let $T : A \to A$. We say that T is a multiplier on A if it satisfies condition

$$(I) \qquad aT(b) = T(ab) = T(a)b, \ (\forall)\ a, b \in A.$$

If T satisfies only the condition

$$(II) \qquad aT(b) = T(a)b, \ (\forall)\ a, b \in A,$$

we say that T is a weak multiplier.

In the paper [11], it is given an example of an operator T, which satisfies the condition (II), but it not satisfies (I). See also [12]. The theory of multipliers (centralizers) on Banach algebra was discussed in now classical papers [4], [9] and [20]. The first who investigated the theory of multipliers in Banach algebras was C. Foiaş, in connection with a certain problem of abstract harmonic analysis. See also the recent papers [6], [13], [19]. See also the book [14].

We say that the algebra A is without the left order if $yA = \{0\}$ implies $y = 0$.

We say that the algebra A is without the right order if $Ay = \{0\}$ implies $y = 0$.

2 Hyperstability of multipliers

In the following, we denote by A a Banach algebra, $T : A \to A$ and $\gamma : A \times A \to [0, \infty)$.

We prove a result on the hyperstability of multipliers. First, we give the following Lemma.

Lemma 2.1. *(Folklore) If $T : A \to A$ is a weak multiplier and A is without right order, then T is a multiplier.*

Proof. Let x be an arbitrary element of A. Since T is a weak multiplier, we have

$$xT(ab) = T(x)ab = xT(a)b.$$

It follows $x[T(ab) - T(a)b] = 0, \ (\forall)\ x \in A$.
Because A is without right order, we obtain $T(ab) = T(a)b = aT(b)$. $\square$

Theorem 2.2. *We suppose that A is a Banach algebra without right order. We suppose that the following conditions holds:*

(i) $\|a(Tb) - (Ta)b\| \le \gamma(a, b)$, *for all* $a, b \in A$;

(ii) $(\exists)(\lambda_n) \subset \mathbb{C} \setminus \{0\}$ *such that* $\lim_{n\to\infty} \dfrac{\gamma(\lambda_n a, b)}{|\lambda_n|} = 0$, *for all* $a, b \in A$.

Then T is a multiplier of A.

Proof. We prove that T is a homogeneous mapping. Let be $\lambda \in \mathbb{C}$ and $x \in A$. In (i), we take $a = \lambda_n x$ and $b = \lambda a$ to obtain:

$$\|\lambda_n x T(\lambda a) - T(\lambda_n x)\lambda a\| \leq \gamma(\lambda_n x, \lambda a)$$

We have

$$\begin{aligned}\|\lambda_n x T(\lambda a) - \lambda_n x \lambda T(a)\| &\leq \|\lambda_n x T(\lambda a) - T(\lambda_n x)\lambda a\| + \|T(\lambda_n x)\lambda a - \lambda_n x \lambda T(a)\| \\ &\leq \gamma(\lambda_n x, \lambda a) + |\lambda| \|T(\lambda_n x)a - \lambda_n x T(a)\| \\ &\leq \gamma(\lambda_n x, \lambda a) + |\lambda|\gamma(\lambda_n x, a)\end{aligned}$$

hence

$$\|xT(\lambda a) - x\lambda T(\lambda)\| \leq \frac{\gamma(\lambda_n a, \lambda a)}{|\lambda_n|} + |\lambda| \frac{\gamma(\lambda_n x, a)}{|\lambda_n|}.$$

We take $n \to \infty$ and use condition (ii) to obtain $x[T(\lambda a) - \lambda T(a)] = 0$.
Since A is without right order we obtain $T(\lambda a) = \lambda T(a)$.
We prove that T is a multiplier. Since T is homogeneous, we have

$$\begin{aligned}\|a(Tb) - (Ta)b\| &= \left\| \frac{1}{\lambda_n}[\lambda_n a T(b) - T(\lambda_n a)b] \right\| \\ &\leq \frac{1}{|\lambda_n|}\gamma(\lambda_n a, b)\end{aligned}$$

We take $n \to \infty$ to obtain $a(Tb) - (Ta)b = 0$. Hence T is a weak multiplier and we apply Lemma 2.1 to obtain that T is a multiplier. □

Corollary 2.3. *We suppose that A is a Banach algebra without right order and $T : A \to A$. We suppose that*

$$\|a(Tb) - (Ta)b\| \leq \gamma(a, b), \ (\forall) \ a, b \in A.$$

We consider fixed $s \in \{1, -1\}$. We suppose that there exists $r \in [0, 2^s)$ such that the following condition holds

$$\gamma(2^s a, b) \leq r \cdot \gamma(a, b).$$

Then T is multiplier.

Proof. We have $\gamma(2^{sn}a, b) \leq r^n \gamma(a, b)$ hence

$$\frac{\gamma(2^{sn}a, b)}{2^{sn}} \leq \left(\frac{r}{2^s}\right)^n \gamma(a, b) \to 0, \text{ as } n \to \infty$$

and we apply Theorem 2.2. □

Remark 2.4. *If we take the particular case when*

$$\gamma(a,b)=\varepsilon\|a\|^p\|b\|^p,$$

with $p\geq 0,\ p\neq 1$, *we obtain the result of Theorem 1.1 in [17].*

For the case $p=1$, we have an extension of a counter example from the paper [17].

Proposition 2.5. *Let* $A\neq\{0\}$ *be a Banach algebra. Then for all* $\varepsilon>0$, *there exists*

$$f:A\to A$$

such that

$$\|af(b)-f(a)b\|\leq\varepsilon\|a\|\|b\|,\ (\forall)\ a,b\in A$$

and f *is not a multiplier.*

Proof. We take $f:A\to A$,

$$f(a)=\frac{\varepsilon\|a\|}{2(1+\|a\|)}a$$

We have

$$\begin{aligned}\|af(b)-f(a)b\|&=\left\|\frac{\varepsilon\|b\|}{2(1+\|b\|)}ab-\frac{\varepsilon\|a\|}{2(1+\|a\|)}ab\right\|\\&\leq\left(\frac{\varepsilon\|b\|}{2(1+\|b\|)}+\frac{\varepsilon\|a\|}{2(1+\|b\|)}\right)\|ab\|\\&\leq\varepsilon\|a\|\|b\|\end{aligned}$$

We prove that there exists $a,b\in A$ such that $af(b)\neq f(a)b$. If $af(b)=f(a)b$, $(\forall)a,b\in A$, we have

$$\frac{\varepsilon\|b\|}{2(1+\|b\|)}ab=\frac{\varepsilon\|a\|}{2(1+\|a\|)}ab.$$

We take norms

$$\frac{\|a\|\|b\|^2}{1+\|b\|}=\frac{\|a\|^2\|b\|}{1+\|a\|}$$

and for $a,b\neq 0$,

$$\frac{\|b\|}{1+\|b\|}=\frac{\|a\|}{1+\|a\|}.$$

It follows $\|b\|+\|b\|\|a\|=\|a\|+\|a\|\|b\|$. Hence $\|a\|=\|b\|$, $(\forall)a,b\neq 0$, which is absurd. $\square$

3 Generalized Hyers-Ulam-Rassias stability of multipliers

Let A be a Banach algebra. Now we do not assume that A is without right order.

Theorem 3.1. *Let A be a Banach algebra, $(\lambda_n) \subset \mathbb{C} \setminus \{0\}$ and $f : A \to A$ be such that*

(iii) $\Phi(a) := \sum_{n=0}^{\infty} \left\| \frac{f(\lambda_{n+1}a)}{\lambda_{n+1}} - \frac{f(\lambda_n a)}{\lambda_n} \right\| < \infty, \ (\forall) a \in A;$

and

(iv) $\|af(b) - f(a)b\| \leq \gamma(a,b), (\forall) a, b \in A$

where $\gamma : A \times A \to [0, \infty)$ is such that

(v) $\lim_{n\to\infty} \frac{\gamma(\lambda_n a, \lambda_n b)}{|\lambda_n|^2} = 0$

If $\lambda_0 = 1$, then there exists a weak multiplier T of A such that

$$\|f(a) - T(a)\| \leq \Phi(a), \ a \in A.$$

Proof. Let a be a fixed element of A. From (iii) it follows hat for all $\varepsilon > 0$, there exists $N(\varepsilon)$ such that

$$\sum_{k=n}^{n+p} \left\| \frac{f(\lambda_{k+1}a)}{\lambda_{k+1}} - \frac{f(\lambda_k a)}{\lambda_k a} \right\| < \varepsilon, \text{ for } n \geq N(\varepsilon), \ p \geq 1.$$

It follows

$$\left\| \frac{f(\lambda_{n+p+1}a)}{\lambda_{n+p+1}} - \frac{f(\lambda_n a)}{\lambda_n} \right\| < \varepsilon.$$

Hence $\left\{ \frac{f(\lambda_n a)}{\lambda_n} \right\}$ is a Cauchy sequence.

Since A is a Banach algebra, we deduce that this sequence converges. We denote by

$$T(a) := \lim_{n\to\infty} \frac{f(\lambda_n a)}{\lambda_n}.$$

We have

$$\|\lambda_n a f(\lambda_n b) - f(\lambda_n a)\lambda_n b\| \leq \gamma(\lambda_n a, \lambda_n b)$$

hence

$$\left\| a\frac{f(\lambda_n b)}{\lambda_n} - \frac{f(\lambda_n a)}{\lambda_n} b \right\| \leq \frac{\gamma(\lambda_n a, \lambda_n b)}{|\lambda_n|^2}$$

We take $n \to \infty$ to obtain

$$\|aT(b) - T(a)b\| = 0$$

hence T is a weak multiplier.

From (iii), we have

$$\sum_{k=0}^{n-1} \left\| \frac{f(\lambda_{k+1}a)}{\lambda_{k+1}} - \frac{f(\lambda_k a)}{\lambda_k} \right\| \leq \Phi(a),\ \ n \geq 1.$$

hence

$$\left\| \frac{f(\lambda_n a)}{\lambda_n} - f(a) \right\| \leq \Phi(a),\ \ n \geq 1. \tag{1}$$

We take $n \to \infty$ to obtain $\|T(a) - f(a)\| \leq \Phi(a),\ \ a \in A.$ □

Corollary 3.2. *Let be $s \in \{1, -1\}$. We suppose that $\psi : [0, \infty) \to [0, \infty)$ and there exists $r \in [0, 2^s)$ such that*

$$\psi(2^s t) \leq r\psi(t),\ \ t \in [0, \infty).$$

We suppose (iv) and (v) take place, where $\gamma : A \times A \to [0, \infty)$ and the following condition holds

(vi) $\|f(2^s a) - 2^s f(a)\| \leq \psi(\|a\|), (\forall)\ a \in A.$

Then there exists a weak multiplier $T : A \to A$ such that

$$\|f(a) - T(a)\| \leq \frac{1}{2^s - r}\psi(\|a\|),\ \ a \in A.$$

From (vi), we have $\|f(2^{s(n+1)}a) - 2f(2^{sn}a)\| \leq r^n \psi(\|a\|)$ hence

$$\Phi(a) := \sum_{n=0}^{\infty} \left\| \frac{f(2^{s(n+1)}a)}{2^{s(n+1)}} - \frac{f(2^{sn}a)}{2^{sn}} \right\| \leq \frac{1}{2^s} \sum_{n=0}^{\infty} \left(\frac{r}{2^s} \right)^n \psi(\|a\|) = \frac{1}{2^s - r}\psi(\|a\|)$$

and we apply Theorem 3.1.

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In memory of Professor Borislav D. CRSTICI on the Centenary of his birth.

CONTRIBUTIONS TO THE STUDY OF THE TURNING CURVE OF A WHEELED TRACTOR

Marius Valentin BOLDEA, Daniel POPA

Abstract

This paper presents an analytical method for determining the turning curve of a wheeled tractor using certain simplifying assumptions. The study addresses the challenges associated with the turning curve, which is significantly influenced by the rotation of the front wheels relative to the longitudinal axis. By formulating the problem analytically, the paper derives equations governing the tractor's movement, providing an intrinsic description of the desired curve. The analytical approach reveals that the turning trajectory consists of two arcs of Cornu's spiral and a circular arc. Comparisons between the graphical method and the proposed analytical solution demonstrate the efficacy and simplicity of the latter. The main parameters of the turning curve are computed directly, facilitating practical applications in agricultural operations.
[1]

1 Introduction

In various agricultural tasks involving the tractor (plowing, sowing, etc.), part of the distance traveled consists of turning curves. The empty moves made during

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turns at the ends of the plots are significant: according to Academician Svirșcevski and Șandru, on a plowed area of 100 hectares, the tractor covers a distance of 570 km, of which 53 km are traveled during turning. On average, the length of these empty moves at the plot ends represents 5-15% of the total distance traveled by the assembly. Thus, it is essential to understand this turning curve to minimize unnecessary moves.

In some specialized treatises (e.g., *Exploitation of the Machinery and Tractor Park* by A. Șandru, I. Niță, and C. Ștefan), the tractor's turning curve is approximated by a circle with a minimum radius. However, in reality, the transition from straight-line travel to circular motion cannot happen instantaneously, as would be assumed for a circular path, but instead takes time, estimated by some authors to be 3-5 seconds or even more under certain conditions. As a result, during turning, the tractor describes a certain curve. Difficulties in accurately determining the curve arise because the turning radius is variable, and the turning center is unstable.

The determination of the turning trajectory of a wheeled tractor has been the subject of research by several scientists. For instance, Academician B.S. Svirșcevski, in his work *Exploitation of the Machinery Park* (p.191), addresses this issue, presenting a construction method for this trajectory developed by Grigoriev and Cuvicinski. Svirșcevski emphasizes the problem's analytical complexity in his book and ultimately provides only the graphical construction method mentioned.

2 Background and Analytical Method

In this paper, using certain simplifying assumptions, an analytical method for determining the turning curve of a wheeled tractor is presented.

It is known that the tractor's turning curve is influenced by the rotation of the front wheels relative to the longitudinal axis. Let's consider a tractor position where the plane of the front wheels is rotated relative to the longitudinal axis by specific angles α' and α'' (see Fig. 1). Given that the tractor moves without slipping, the turning center is determined by the extensions of the front and rear wheels.

The plane XX' perpendicular to OB represents the plane of a third front wheel that could replace the two wheels to allow the tractor to follow the same path. The plane XX' makes an angle α with the longitudinal axis, located between α' and α''. Practically, α' and α'' can be considered equal to α.

Point A, whose trajectory is to be determined, is called the assembly center. From triangle ΔOAB and the notations in Fig. 1, the instantaneous curvature radius is written as:

$$R = \frac{L}{\tan\alpha}. \quad (1)$$

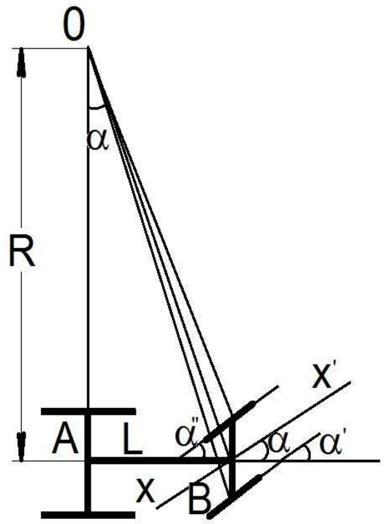

Figure 1: Turning curve geometry

Given that α does not take large values, we can replace $\tan\alpha$ with α, and equation (1) becomes:

$$R = \frac{L}{\alpha}. \quad (2)$$

3 Uniform Rotation of the Steering Wheel

Assuming the steering wheel rotates uniformly:

$$\alpha = \omega\tau, \quad (3)$$

where ω is the angular rotation speed of plane XX' (proportional to the angular rotation speed of the steering wheel), and τ represents time. We also have:

$$s = v\tau, \quad (4)$$

where v is the tractor's speed. Considering equations (3) and (4), equation (2) becomes:

$$R = \frac{Lv}{\omega s}. \quad (5)$$

Equation (5) is, in fact, the intrinsic equation of the desired curve.

$$\frac{1}{R} = \frac{d\varphi}{ds}.$$

4 Curvature Definition and Integration

Starting from the definition of curvature:

$$\varphi = \frac{\omega}{2Lv} s^2, \quad (6)$$

where φ is the angle between the OX axis and the tangent at a point on the curve. Using the direction cosines definition:

$$\frac{dx}{ds} = \cos\varphi, \quad \frac{dy}{ds} = \sin\varphi,$$

by integrating, we derive the equation of the curve in parametric form:

$$x = \int_0^s \cos\left(\frac{\omega}{2Lv} s^2\right) ds, \quad 0 < s < s_1 \quad (7)$$

$$y = \int_0^s \sin\left(\frac{\omega}{2Lv} s^2\right) ds.$$

or, after substitution:

$$s = kt, \quad (8)$$

where:

$$k = \sqrt{\frac{\pi L v}{\omega}}. \quad (9)$$

5 Fresnel Integrals and Cornu's Spiral

The relationships in (7) become:

$$x = k \int_0^t \cos\left(\frac{\pi}{2} t^2\right) dt \quad 0 < t < t_1. \quad (10)$$

$$y = k \int_0^t \sin\left(\frac{\pi}{2} t^2\right) dt$$

These expressions highlight the Fresnel integrals, which are present and have been tabulated. These equations are those of Cornu's spiral.

In fact, equations (10) represent only a portion of the turning curve, specifically the section during steering wheel rotation, up to the point where parameter s reaches $s_1 = v\tau_1\lambda r_1$. After the steering wheel reaches its maximum rotation, the tractor (more precisely, point A) describes an arc of a circle, after which, by reversing the steering wheel rotation, it follows a return trajectory.

6 Symmetry of the Turning Curve

During the return phase, the curvature radius is still given by equation (2), except that angle α decreases over time,

$$\alpha = \alpha_{\max} - \omega\tau,$$

so the intrinsic equation becomes:

$$R' = \frac{Lv}{\omega\left(\frac{\alpha_{\max} v}{\omega} - s\right)}.$$

Switching to Cartesian coordinates as in the initial part of the curve, ignoring translations and rotations, we obtain:

$$x' = \int_0^{s'} \cos\left(\frac{\omega}{2Lv} s'^2\right) dc', \quad y' = \int_0^{s'} \sin\left(\frac{\omega}{2Lv} s'^2\right) dc'. \quad (11)$$

Comparing equations (7) and (11), we find that the two curves differ only by a symmetry. Thus, to trace the tractor's turning curve (point A) by 180°, which consists of two arcs of Cornu's spiral and a circular arc, the OM_1 portion is traced through points using relationships (10) (Fig. 2), with coordinates of point $M_1(x_1, y_1)$ corresponding to parameter $s_1 = v\tau_1$. The tangent at M_1, which actually makes an angle φ_1 with the OX axis as given by equation (6), helps locate the curvature center O_1 using the curvature radius at point M_1 from equation (1). The parallel to the OX axis passing through O_1 is the symmetry axis of the turning curve.

In Fig. 2, the coordinate axes are chosen such that the constants resulting from the transition from intrinsic equation (5) to Cartesian coordinates (7) are zero.

In the case of a 90° turn, we get the situation in Fig. 3, where the symmetry axis is parallel to the secondary bisector passing through point O_1 (Fig. 3).

7 Comparison with Graphical Method

To see the correspondence of these theoretical considerations, we can compare the turning curve of a tractor drawn according to the graphical trajectory of Grigoriev and Cuvicinski with the trajectory built analytically under the same conditions, following the above-described method. In Fig. 4, the curve is drawn using the graphical method, while the points marked with circles represent the analytical coordinates calculated using equations (10). It is observed that, practically, both methods yield the same curve.

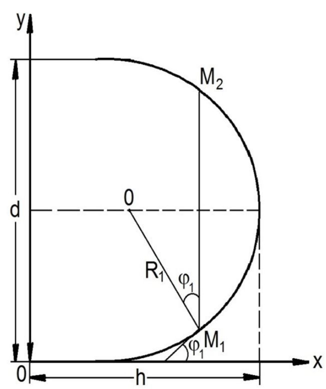

Figure 2: Calculated trajectory

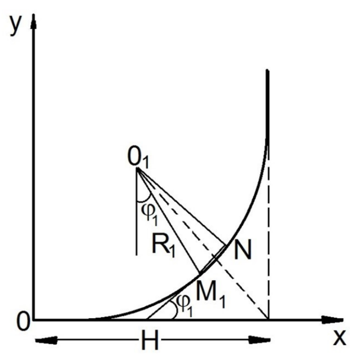

Figure 3: Symmetry of the Curve

Compared to the graphical method, the analytical method, specifically the construction based on equations (10), is simpler, as the values of the integrals can be found in tables.

The construction and calculation of coordinates were performed under the following initial conditions:

$$\tau = 5\,\mathrm{s}, \quad L = 2.445\,\mathrm{m}, \quad v = 1.2\,\mathrm{m/s}, \quad \omega = 6\,\mathrm{rad/s}.$$

The analytical method also allows for the direct calculation of the main parameters of the turning curve d and h (Fig. 2) and H (Fig. 3), without needing graphical construction. Indeed, from Fig. 2:

$$d = 2(y_1 + R_1 \cos\varphi_1),$$

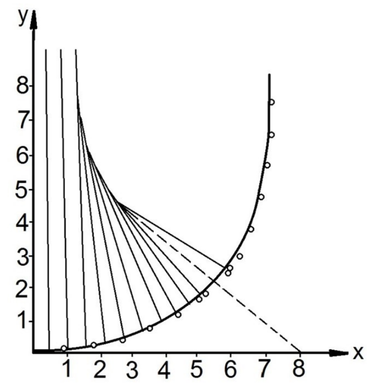

Figure 4: Calculated Trajectory and Graphical Method

$$h = x_1 + R_1 - R_1 \sin\varphi_1,$$

or considering relations (10):

$$d = 2k \int_0^{t_1} \sin\left(\frac{\pi}{2}t^2\right) dt + 2R_1 \cos\varphi_1, \quad (12)$$

$$h = k \int_0^{t_1} \cos\left(\frac{\pi}{2}t^2\right) dt + R_1(1 - \sin\varphi_1), \quad (13)$$

where:

$$k = \sqrt{\frac{\pi L v}{\omega}}, \quad s_1 = v\tau_1, \quad t_1 = \frac{s_1}{k}, \quad R_1 = \frac{L}{\tan\alpha_{\max}}, \quad \varphi_1 = \frac{\omega}{2Lv}s_1^2.$$

In the 90° turning case, Fig. 3 yields:

$$M_1N = 2R_1 \sin\left(\frac{\pi}{4} - \varphi_1\right),$$

and so:

$$H = x_1 + 2R_1 \sin\left(\frac{\pi}{4} - \varphi_1\right)\cos\frac{\pi}{4} + y_1,$$

or considering relations (10):

$$H = k \int_0^{t_1} \cos\left(\frac{\pi}{2}t^2\right) dt + k \int_0^{t_1} \sin\left(\frac{\pi}{2}t^2\right) dt + \sqrt{2}R_1 \sin\left(\frac{\pi}{4} - \varphi_1\right), \quad (14)$$

where k, R_1, t_1, and φ_1 are calculated similarly as for relations (12) and (13).

In conclusion, the turning curve of a wheeled tractor can be represented using equations (10), while the main parameters d, h for a 180° turn, and H for a 90° turn, can be calculated using equations (12), (13), and (14).

In the particular case $\tau = 0$ where the tractor enters and exits the turn with the steering wheel rotated and fixed, the primary parameters given by formulas (12), (13), and (14) take the values $d = 2R_1$, $h = R_1$, and $H = R_1$, which is evident.

For the numerical case considered in curve construction:

$$\tau = 5\,\mathrm{s}, \quad L = 2.445\,\mathrm{m}, \quad v = 1.2\,\mathrm{m/s}, \quad \omega = 6\,\mathrm{rad/s},$$

the intermediate constants are:

$$k = 9.38, \quad s_1 = 6, \quad t_1 = 0.64, \quad R_1 = 4.23,$$

while the main parameters derived from equations (12), (13), and (14) are:

$$d = 9.27\,\mathrm{m}, \quad h = 7.45\,\mathrm{m}, \quad H = 7.85\,\mathrm{m}.$$

Conclusions

The study of the turning curve of a wheeled tractor reveals the complex dynamics involved in transitioning from straight-line motion to a curved path. Using simplifying assumptions, an analytical approach was developed that accurately models the tractor's turning trajectory. The results demonstrate that the turning path can be described by two arcs of Cornu's spiral and a circular segment, providing a more realistic representation than simple circular approximations.

The proposed analytical method has several advantages over traditional graphical methods, including simplified calculations and direct determination of key parameters such as the curvature radius and angular relationships. These findings have practical implications for optimizing tractor operations in agricultural settings, reducing unnecessary movement, and enhancing efficiency.

Furthermore, comparisons with graphical methods indicate a high level of agreement, validating the analytical model. The equations derived, particularly those involving intrinsic and Cartesian coordinates, offer a robust foundation for further research and practical applications in machinery design and agricultural field management.

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In memory of Professor Borislav D. CRSTICI on the Centenary of his birth.

DYNAMICS OF a FIVE-DIMENSIONAL MATHEMATICAL MODEL FOR UNEMPLOYMENT

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Abstract

The purpose of this paper is to develop and analyse a five-dimensional mathematical model of labor market slack, incorporating both unemployment and employment characterised by limited working hours.
[1]

1 Introduction

Over time, one of the significant challenges faced by a country is economic downturn driven by unemployment. The causes of this phenomenon are numerous and vary depending on the specific context of each country. Key factors contributing to the uncontrolled rise in unemployment include issues such as population growth and sluggish economic expansion.

Previous research in the field has explored various aspects of five-dimensional mathematical models to control the unemployment, including studies involving time delays [10], [11].

Distributed time delays have been widely applied in mathematical models across different disciplines, including population biology, epidemiology, and economics [1], [2], [3], [4], [5], [6], [7], [9].

In this paper, we analyze the nonlinear unemployment model without considering time delays, utilizing the stability properties of differential equations to examine its dynamics.

[1]

Keywords and phrases: *unemployment model, equilibrium points, local stability, global stability*

2 The mathematical model

The following five variables which describe the mathematical model for to control the unemployment are considered: the number of unemployed persons $X_1(t)$, the number of immigrants $X_2(t)$, the number of temporary employed persons $X_3(t)$, the number of regularly employed persons $X_4(t)$, respectively and the number of available vacancies $X_5(t)$ at time t.

The mathematical model proposed is:

$$\begin{cases} \dot{X}_1(t) = a_1 - a_2X_1(t)X_5(t) + a_3X_4(t) - a_4X_1(t) + a_5X_3(t) - b_1X_1(t), \\ \dot{X}_2(t) = m_1 - m_2X_2(t)X_5(t) - b_2X_2(t), \\ \dot{X}_3(t) = a_4X_1(t) - a_5X_3(t) - c_1X_3(t)X_5(t) - b_3X_3(t), \\ \dot{X}_4(t) = a_2X_1(t)X_5(t) + m_2X_2(t)X_5(t) + c_1X_3(t)X_5(t) - a_3X_4(t) - b_4X_4(t), \\ \dot{X}_5(t) = c_2X_4(t) - b_5X_5(t), \end{cases} \tag{1}$$

where: a_1 represents the constant growth rate of unemployed persons entering the labor market, a_2 is the rate of hiring, a_3 describes the rate of firing, a_4 expresses the rate of move to unemployment, a_5 - the rate of move to temporary employment; b_1 is the rate of migration of unemployed persons; b_2 - the rate of return or death, b_3 describes the rate of temporary employment, b_4 represents the rate of retirement, migration or death of employed persons; b_5 - the rate of available vacancies; c_1 is the rate of upgrading from limited hours to regular employment; c_2 is the rate of vacancies creation in response to current employment conditions; m_1 represents the exogenous increase in migration and m_2 is the migrants' entry rate into regular employment.

Considering the previously mentioned factors, we present a schematic representation of the model, as depicted in Figure 1.

3 Adimensional model

In order to write the adimensional system we consider the changes of variables:

$$\begin{cases} x_1(t) = \dfrac{a_2c_2}{a_5^2}X_1\left(\dfrac{t}{a_5}\right), \\ x_2(t) = \dfrac{a_2c_2}{a_5^2}X_2\left(\dfrac{t}{a_5}\right), \\ x_3(t) = \dfrac{a_2c_2}{a_5^2}X_3\left(\dfrac{t}{a_5}\right), \\ x_4(t) = \dfrac{a_2c_2}{a_5^2}X_4\left(\dfrac{t}{a_5}\right), \\ x_5(t) = \dfrac{a_2}{a_5}X_5\left(\dfrac{t}{a_5}\right), \end{cases} \tag{2}$$

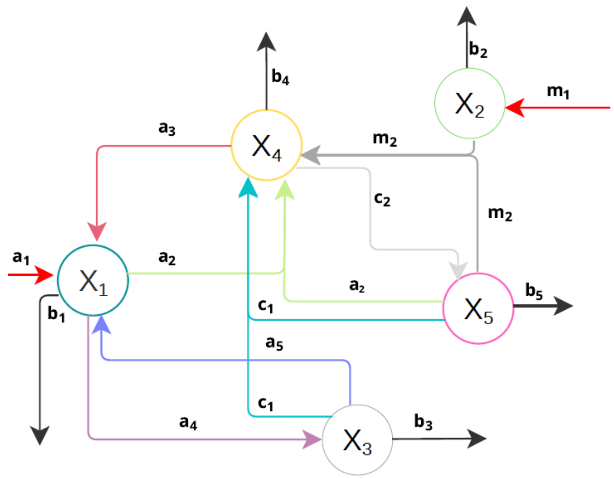

Figure 1: Representation of the five-dimensional model without time delays given in (1).

.

and the system (1) becomes:

$$\begin{cases} \dot{x}_1(t) = \gamma_1 - x_1(t)x_5(t) + \alpha_3 x_4(t) - \alpha_4 x_1(t) + x_3(t) - \beta_1 x_1(t), \\ \dot{x}_2(t) = \gamma_2 - \alpha_2 x_2(t)x_5(t) - \beta_2 x_2(t), \\ \dot{x}_3(t) = \alpha_4 x_1(t) - x_3(t) - \alpha_1 x_3(t)x_5(t) - \beta_3 x_3(t), \\ \dot{x}_4(t) = x_1(t)x_5(t) + \alpha_2 x_2(t)x_5(t) + \alpha_1 x_3(t)x_5(t) - \alpha_3 x_4(t) - \beta_4 x_4(t), \\ \dot{x}_5(t) = x_4(t) - \beta_5 x_5(t), \end{cases} \tag{3}$$

where the coefficients are expressed as:

$$\begin{aligned} &\gamma_1 = \frac{a_1 a_2 c_2}{a_5^3}, \quad \gamma_2 = \frac{a_2 c_2 m_1}{a_5^3}, \\ &\alpha_1 = \frac{c_1}{a_2}, \quad \alpha_2 = \frac{m_2}{a_2}, \quad \alpha_3 = \frac{a_3}{a_5}, \quad \alpha_4 = \frac{a_4}{a_5}, \\ &\beta_1 = \frac{b_1}{a_5}, \quad \beta_2 = \frac{b_2}{a_5}, \quad \beta_3 = \frac{b_3}{a_5}, \quad \beta_4 = \frac{b_4}{a_5}, \quad \beta_5 = \frac{b_5}{a_5}, \end{aligned}$$

4 Equilibrium points of the model

As constant solutions of system (3), the equilibrium points satisfy the following algebraic equations:

$$\begin{cases} \gamma_1 - x_1x_5 + \alpha_3x_4 - \alpha_4x_1 + x_3 - \beta_1x_1 = 0, \\ \gamma_2 - \alpha_2x_2x_5 - \beta_2x_2 = 0, \\ \alpha_4x_1 - x_3 - \alpha_1x_3x_5 - \beta_3x_3 = 0, \\ x_1x_5 + \alpha_2x_2x_5 + \alpha_1x_3x_5 - \alpha_3x_4 - \beta_4x_4 = 0, \\ x_4 - \beta_5x_5 = 0. \end{cases} \tag{4}$$

On the one hand, we note that system (4) has at least one solution, which corresponds to the case $x_5 = 0$. As a result, we obtain the equilibrium point S^0:

$$S^0 := (\delta_1(1+\beta_3), \delta_2, \delta_1\alpha_4, 0, 0)\,,$$

with

$$\delta_1 = \frac{\gamma_1}{\beta_1 + \beta_1\beta_3 + \alpha_4\beta_3} \quad \text{and} \quad \delta_2 = \frac{\gamma_2}{\beta_2}.$$

This equilibrium point represents the situation in which there is neither regular employment and no available vacancies.

Now we introduce the basic reproduction number R_0. It serves as a threshold parameter that predicts whether or not problems with immigration, unemployment, and temporary employment will increase or decrease. We determine R_0 using the next-generation matrix method:

$$R_0 = \frac{\delta_1(1+\beta_3+\alpha_1\alpha_4) + \delta_2\alpha_2}{(\alpha_3+\beta_4)\beta_5}.$$

On the other hand, if and only if x_5 is the solution of the following cubic equation, system (4) has at least one solution with the last component $x_5 > 0$:

$$E_3x_5^3 + E_2x_5^2 + E_1x_5 + E_0 = 0, \tag{5}$$

with the coefficients

$$\begin{aligned} E_3 &= \alpha_1\beta_4\beta_5, \\ E_2 &= \alpha_1(\nu_2\beta_4\beta_5 - \gamma_1 - \gamma_2) + \beta_4\beta_5(1+\beta_3+\alpha_1\alpha_4) + \alpha_1\beta_1\mu, \\ E_1 &= (\nu_2\beta_4\beta_5 - \gamma_1 - \gamma_2)(1+\beta_3+\alpha_1\alpha_4) - \nu_2\gamma_1\alpha_1 + \mu\beta_1\left(1+\beta_3+\alpha_1\alpha_4\frac{\nu_1}{\beta_1}\right) \\ &\quad + (\mu\nu_2 - \gamma_2)\beta_1\alpha_1, \\ E_0 &= \frac{\mu\gamma_1\gamma_2}{\alpha_2\delta_1\delta_2}(1-R_0), \end{aligned}$$

where R_0, δ_1 and δ_2 are given above and

$$\nu_1 = \frac{\beta_3}{\alpha_1}, \qquad \nu_2 = \frac{\beta_2}{\alpha_2}, \qquad \mu = (\alpha_3 + \beta_4)\beta_5.$$

System (3) in this case has at least one equilibrium point S^+ with the form:

$$S^+ := \left(-\frac{Q(x_5) + \nu_1 d(x_5)}{(\beta_1 - \nu_1)(x_5 + \nu_2)}, \frac{\gamma_2}{\alpha_2(\nu_2 + x_5)}, \frac{Q(x_5) + \beta_1 d(x_5)}{\alpha_1(\beta_1 - \nu_1)(\nu_2 + x_5)}, \beta_5 x_5, x_5 \right),$$

where

$$\begin{aligned} Q(x_5) &= (\beta_4\beta_5 x_5 - \gamma_1 - \gamma_2)(x_5 + \nu_2) + \gamma_2\nu_2, \\ d(x_5) &= \mu(x_5 + \nu_2) - \gamma_2. \end{aligned}$$

We actually distinguish between two cases:

Case 1: If $R_0 > 1$ then $E_0 < 0$ and system (3) has at least one positive equilibrum point. Moreover, if either $E_1 < 0$ or $E_1 > 0$ and $E_2 > 0$, Descartes' rule of signs guarantees the existence of a unique positive equilibrium point S^+.

Case 2: If $R_0 < 1$ then $E_0 > 0$, we may have either two or zero positive equilibrium points. For instance, if $E_1 > 0$ and $E_2 > 0$, system (3) does not have positive equilibrium points.

5 Positivity and boundedness of solutions

Theorem 5.1. *The open positive (nonnegative) orthant $\mathbb{R}^5_+ = (0, \infty)^5$ is invariant to the flow of system* (3)*. Furthermore, denoting $\beta_m = \min(\beta_1, \beta_2, \beta_3, \beta_4)$, the set*

$$\Omega = \left\{ (x_1, x_2, x_3, x_4, x_5) : 0 \leq x_1 + x_2 + x_3 + x_4 \leq \frac{\gamma_1 + \gamma_2}{\beta_m},\ 0 \leq x_5 \leq \frac{\gamma_1 + \gamma_2}{\beta_m \beta_5} \right\}$$

is a region of attraction for the system (1)*, attracting all the solutions with initial conditions in the open positive orthant $\mathbb{R}^5_+$.*

Proof. The positivity of solutions is proved using Theorem 3.4. from [12].

For the boundedness of solutions, from the first four equations of system (3) we observe that

$$\frac{d}{dt}[x_1(t) + x_2(t) + x_3(t) + x_4(t)] = \gamma_1 + \gamma_2 - \beta_1 x_1(t) - \beta_2 x_2(t) - \beta_3 x_3(t) - \beta_4 x_4(t),$$

and taking into account that $\beta_m = \min(\beta_1, \beta_2, \beta_3, \beta_4)$ we have

$$\frac{d}{dt}[x_1(t)+x_2(t)+x_3(t)+x_4(t)] \leq \gamma_1+\gamma_2-\beta_m[x_1(t)+x_2(t)+x_3(t)+x_4(t)], \quad \forall\, t > 0.$$

Therefore, using basic differential inequality techniques, we obtain:

$$x_1(t)+x_2(t)+x_3(t)+x_4(t) \leq e^{-\beta_m t}\left(x_1(0)+x_2(0)+x_3(0)+x_4(0)-\frac{\gamma_1+\gamma_2}{\beta_m}\right)+\frac{\gamma_1+\gamma_2}{\beta_m},$$

for any $t \geq 0$.

On one hand, if $x_1(0)+x_2(0)+x_3(0)+x_4(0) \leq \frac{\gamma_1+\gamma_2}{\beta_m}$ it follows that $(x_1(0)+x_2(0)+x_3(0)+x_4(0) \leq \frac{\gamma_1+\gamma_2}{\beta_m}$, for any $t > 0$. Otherwise, if $x_1(0)+x_2(0)+x_3(0)+x_4(0) > \frac{\gamma_1+\gamma_2}{\beta_m}$, we have

$$\limsup_{t\to\infty}[x_1(t)+x_2(t)+x_3(t)+x_4(t)] \leq \frac{\gamma_1+\gamma_2}{\beta_m}.$$

Moreover, the last equation of (3) gives

$$\frac{d}{dt}[x_5(t)] = x_4(t) - \beta_5 x_5(t) \leq \frac{\gamma_1+\gamma_2}{\beta_m} - \beta_5 x_5(t).$$

Applying the generalized l'Hospital rule, we deduce:

$$\limsup_{t\to\infty} x_5(t) \leq \frac{\gamma_1+\gamma_2}{\beta_m\beta_5},$$

and the desired conclusion is proved.

□

Remark 5.2. *The previous theorem states that solutions of system* (3) *originating from initial conditions belonging to the open positive orthant of* $\mathbb{R}^5$*, i.e.* $\varphi_i : (-\infty, 0] \to (0,\infty)$*, for* $i = \overline{1,5}$*, remain positive for any* $t > 0$*. Moreover, such solutions satisfy the inequalities:*

$$\limsup_{t\to\infty}[x_1(t)+x_2(t)+x_3(t)+x_4(t)] \leq \frac{\gamma_1+\gamma_2}{\beta_m} \quad \textit{and} \quad \limsup_{t\to\infty} x_5(t) \leq \frac{\gamma_1+\gamma_2}{\beta_5\beta_m}.$$

5.1 Local stability analysis for S^0

Theorem 5.3. *The equilibrium point* S^0 *of system (3) is locally asymptotically stable if and only if* $R_0 < 1$.

Proof. The characteristic equation is:

$(\lambda+\beta_2)[\lambda^2+\lambda(\beta_3+\beta_1+\alpha_4+1)+\alpha_4\beta_3+\beta_1\beta_3+\beta_1][\lambda^2+\lambda(\alpha_3+\beta_3+\beta_5)$ $+\beta_5(\alpha_3+\beta_3)(1-R_0)] = 0$.

We observe that $\lambda = -\beta_2 < 0$ is a root of this characteristic equation.

The first quadratic polynomial has all the roots with negative real part because all the coefficients are positive.

For the second quadratic polynomial, based on the Routh-Hurwitz conditions, we obtain that all the roots have negative real part if and only if $R_0 < 1$. □

5.2 Global stability analysis for S^0

Theorem 5.4. *The equilibrium point S^0 is globally asymptotically stable in the open positive orthant $\mathbb{R}^5_+$, if the following inequality holds:*

$$R_0 < \frac{\beta_4}{\alpha_3 + \beta_4}. \tag{6}$$

Proof. Considering an arbitrary solution $x_i(t)$, $i = \overline{1,5}$, of system (3), with initial conditions $\varphi_i : (-\infty, 0] \to (0, \infty)$. Using Theorem 5.1 it follows that $x_i(t) > 0$, for any $t > 0$ and $i = \overline{1,5}$. Consequently, we consider the functions:

$$L_k(t) = x_k(t) - x_k^0 - x_k^0 \ln \frac{x_k(t)}{x_k^0}\ ,\ k = \overline{1,5}\ ,$$

and we denote:

$$E_k(t) = 1 - \frac{x_k(t)}{x_k^0} + \ln \frac{x_k(t)}{x_k^0} < 0, \qquad \tilde{E}_k(t) = 1 - \frac{x_k^0}{x_k(t)} + \ln \frac{x_k^0}{x_k(t)} < 0\ ,\ k = \overline{1,5}.$$

Now, we have:

$$\begin{aligned}
L_1'(t) &= \dot{x}_1\Big(1 - \frac{x_1^0}{x_1}\Big) = \Big(1 - \frac{x_1^0}{x_1}\Big)[\gamma_1 - x_1x_5 + \alpha_3x_4 + x_3 - (\alpha_4 + \beta_1)x_1] \\
&= \Big(1 - \frac{x_1^0}{x_1}\Big)[x_1^0x_5^0 - \alpha_3x_4^0 - x_3^0 + (\alpha_4 + \beta_1)x_1^0 - x_1x_5 + \alpha_3x_4 + x_3 - (\alpha_4 + \beta_1)x_1] \\
&= \Big(1 - \frac{x_1^0}{x_1}\Big)\Big[(\alpha_4 + \beta_1)x_1^0\Big(1 - \frac{x_1}{x_1^0}\Big) - x_3^0\Big(1 - \frac{x_3}{x_3^0}\Big) - x_1x_5 + \alpha_3x_4\Big] \\
&= (\alpha_4 + \beta_1)x_1^0\left(2 - \frac{x_1^0}{x_1} - \frac{x_1}{x_1^0}\right) - x_3^0\left(1 + \frac{x_1^0}{x_1} \cdot \frac{x_3}{x_3^0} - \frac{x_3}{x_3^0} - \frac{x_1^0}{x_1}\right) \\
&\quad - x_1x_5 + \alpha_3x_4 - \alpha_3x_4\frac{x_1^0}{x_1} + x_5x_1^0 \\
&\le (\alpha_4 + \beta_1)x_1^0\left(2 - \frac{x_1^0}{x_1} - \frac{x_1}{x_1^0}\right) - x_3^0\left(2 + \ln\frac{x_1^0}{x_1} + \ln\frac{x_3}{x_3^0} - \frac{x_3}{x_3^0} - \frac{x_1^0}{x_1}\right) \\
&\quad - x_1x_5 + \alpha_3x_4 + x_5x_1^0 \\
&= (\alpha_4 + \beta_1)x_1^0(E_1 + \tilde{E}_1) - x_3^0(\tilde{E}_1 + E_3) - x_1x_5 + x_1^0x_5 + \alpha_3x_4 - \alpha_3x_4\frac{x_1^0}{x_1}.
\end{aligned}$$

In addition, we compute:

$$L_2'(t) = \dot{x}_2\left(1 - \frac{x_2^0}{x_2}\right) = \left(1 - \frac{x_2^0}{x_2}\right)(\gamma_2 - \alpha_2x_2x_5 - \beta_2x_2)$$

$$
\begin{aligned}
&= \left(1-\frac{x_2^0}{x_2}\right)(\alpha_2 x_2^0 x_5^0 + \beta_2 x_2^0 - \alpha_2 x_2 x_5 - \beta_2 x_2)\\
&= \beta_2 x_2^0 \left(2-\frac{x_2^0}{x_2}-\frac{x_2}{x_2^0}\right) - \alpha_2 x_2 x_5 + \alpha_2 x_2^0 x_5\\
&= \beta_2 x_2^0 (E_2 + \tilde{E}_2) - \alpha_2 x_2 x_5 + \alpha_2 x_2^0 x_5,
\end{aligned}
$$

$$
\begin{aligned}
L_3'(t) &= \dot{x}_3 \left(1-\frac{x_3^0}{x_3}\right) = \left(1-\frac{x_3^0}{x_3}\right)(\alpha_4 x_1 - x_3 - \alpha_1 x_3 x_5 - \beta_3 x_3)\\
&= \left(1-\frac{x_3^0}{x_3}\right)\left[\alpha_4(x_1 - x_1^0) - (1+\beta_3)(x_3 - x_3^0) - \alpha_1(x_3x_5 - x_3^0 x_5^0)\right]\\
&= \left(1-\frac{x_3^0}{x_3}\right)\left[\alpha_4 x_1^0 \left(\frac{x_1}{x_1^0}-1\right) - x_3^0(1+\beta_3)\left(\frac{x_3}{x_3^0}-1\right) - \alpha_1 x_3 x_5\right]\\
&= -\alpha_4 x_1^0 \left(1+\frac{x_3^0}{x_3}\cdot\frac{x_1}{x_1^0} - \frac{x_3^0}{x_3} - \frac{x_1}{x_1^0}\right) + x_3^0(1+\beta_3)\left(2-\frac{x_3}{x_3^0}-\frac{x_3^0}{x_3}\right)\\
&- \alpha_1 x_3 x_5 + \alpha_1 x_3^0 x_5\\
&\le -\alpha_4 x_1^0 \left(2+\ln\frac{x_3^0}{x_3} + \ln\frac{x_1}{x_1^0} - \frac{x_3^0}{x_3} - \frac{x_1}{x_1^0}\right) + x_3^0(1+\beta_3)\left(2-\frac{x_3}{x_3^0}-\frac{x_3^0}{x_3}\right)\\
&- \alpha_1 x_3 x_5 + \alpha_1 x_3^0 x_5\\
&= -\alpha_4 x_1^0 (E_1 + \tilde{E}_3) + (1+\beta_3) x_3^0 (E_3 + \tilde{E}_3) - \alpha_1 x_3 x_5 + \alpha_1 x_3^0 x_5.
\end{aligned}
$$

Moreover, considering

$$L_{4,5}(t) = x_4 + c x_5,$$

we have

$$
\begin{aligned}
L_{4,5}'(t) &= \dot{x}_4 + c\dot{x}_5\\
&= x_1 x_5 + \alpha_2 x_2 x_5 + \alpha_1 x_3 x_5 - (\alpha_3 + \beta_4 - c) x_4 - c\beta_5 x_5.
\end{aligned}
$$

Further, considering

$$L(t) = L_1(t) + L_2(t) + L_3(t) + L_{4,5}(t)$$

we compute

$$
\begin{aligned}
L'(t) &\le (\alpha_4+\beta_1) x_1^0 (E_1 + \tilde{E}_1) - x_3^0(\tilde{E}_1 + E_3) - x_1 x_5 + x_1^0 x_5 + \alpha_3 x_4\\
&+ \beta_2 x_2^0 (E_2 + \tilde{E}_2) - \alpha_2 x_2 x_5 + \alpha_2 x_2^0 x_5\\
&- \alpha_4 x_1^0 (E_1 + \tilde{E}_3) + (1+\beta_3) x_3^0 (E_3 + \tilde{E}_3) - \alpha_1 x_3 x_5 + \alpha_1 x_3^0 x_5\\
&+ x_1 x_5 + \alpha_2 x_2 x_5 + \alpha_1 x_3 x_5 - (\alpha_3 + \beta_4 - c) x_4 - c\beta_5 x_5\\
&= \beta_1 x_1^0 E_1 + \left[(\alpha_4+\beta_1) x_1^0 - x_3^0\right]\tilde{E}_1 + \beta_2 x_2^0 (E_2 + \tilde{E}_2) + \beta_3 x_3^0 E_3
\end{aligned}
$$

$$+\left[(1+\beta_3)x_3^0-\alpha_4x_1^0\right]\tilde{E}_3-(\beta_4-c)x_4-(c\beta_5-x_1^0-\alpha_2x_2^0-\alpha_1x_3^0)x_5$$
$$=\beta_1x_1^0E_1+\beta_2x_2^0(E_2+\tilde{E}_2)+\beta_3x_3^0E_3-(\beta_4-c)x_4-\beta_5\left[c-R_0(\alpha_3+\beta_4)\right]x_5.$$

Due to the fact that the inequality $R_0 < \frac{\beta_4}{\alpha_3+\beta_4}$ is true, there is $c > 0$ such that $L'(t) < 0$. We conclude that the equilibrium point S^0 of the system (3) is globally asymptotically stable in $\mathbb{R}_+^5$ using LaSalle's invariance principle [8], [12]. □

5.3 Local stability analysis for S^+

The characteristic equation is:

$$P_0(\lambda)+P_1(\lambda)+P_2(\lambda)+P_{12}(\lambda)=0 \tag{7}$$

where

$$\begin{aligned}
P_0(\lambda) &= -(\beta_5+\lambda)(\beta_2+\alpha_2x_5+\lambda)\{(\beta_4+\lambda)[x_5+\lambda+(\alpha_4+x_5+\lambda)\\
&\quad (\beta_3+\alpha_1x_5+\lambda)]+\alpha_3[\alpha_4(\beta_3+\lambda)+\lambda(1+\beta_3+\alpha_1x_5+\lambda)]\}\\
P_1(\lambda) &= -\beta_1(\alpha_3+\beta_4+\lambda)(\beta_5+\lambda)(1+\beta_3+\alpha_1x_5+\lambda)(\beta_2+\alpha_2x_5+\lambda)\\
P_2(\lambda) &= \alpha_2^2\alpha_4x_2x_5(\beta_3+\alpha_1x_5+\lambda)-\alpha_2^2x_2x_5(1+\beta_3+\alpha_1x_5+\lambda)(\lambda+x_5)\\
&\quad -\alpha_4\beta_5(\alpha_3+\beta_4)(\beta_2+\alpha_2x_5+\lambda)-\alpha_1\alpha_4(\beta_2+\alpha_2x_5+\lambda)(x_1x_5+\alpha_1x_3x_5)\\
&\quad +\beta_5(\alpha_3+\beta_4)(1+\beta_3+\alpha_1x_5+\lambda)(\beta_2+\alpha_2x_5+\lambda)(\alpha_4+x_5+\lambda)\\
&\quad -x_5(\beta_2+\alpha_2x_5+\lambda)[\alpha_1x_3+x_1(1+\beta_3+\alpha_1x_5+\lambda)]\\
&\quad -\alpha_1^2x_3x_5(\beta_2+\alpha_2x_5+\lambda)(x_5+\lambda)\\
P_{12}(\lambda) &= \beta_1[-\alpha_2^2x_2x_5(1+\beta_3+\alpha_1x_5+\lambda)-\alpha_1^2x_3x_5(\beta_2+\alpha_2x_5+\lambda)\\
&\quad +\beta_5(\alpha_3+\beta_4)(1+\beta_3+\alpha_1x_5+\lambda)(\beta_2+\alpha_2x_5+\lambda)]
\end{aligned}$$

The algebraic equation of fifth degree for the positive equilibrium point is:

$$\lambda^5+a_{45}\lambda^4+a_{35}\lambda^3+a_{25}\lambda^2+a_{15}\lambda+a_{05}=0, \tag{8}$$

where the coefficients are:

$$\begin{aligned}
a_{45} &= 1+\alpha_3+\alpha_4+\beta_1+\beta_2+\beta_3+\beta_4+\beta_5+(1+\alpha_1+\alpha_2)x_5\\
a_{35} &= \beta_2+\alpha_4\beta_2+\alpha_4\beta_3+\beta_2\beta_3+\beta_4+\alpha_4\beta_4+\beta_2\beta_4+\beta_3\beta_4+\beta_5+\alpha_4\beta_5+\beta_2\beta_5\\
&\quad +\beta_3\beta_5+[1+\beta_2+\beta_3+\beta_4+\beta_5+\alpha_1(\alpha_4+\beta_2+\beta_4+\beta_5)\\
&\quad +\alpha_2(1+\alpha_4+\beta_3+\beta_4+\beta_5)]x_5+(\alpha_1+\alpha_2+\alpha_1\alpha_2)x_5^2\\
&\quad +\alpha_3[1+\alpha_4+\beta_1+\beta_2+\beta_3+(\alpha_1+\alpha_2)x_5]\\
&\quad +\beta_1[1+\beta_2+\beta_3+\beta_4+\beta_5+(\alpha_1+\alpha_2)x_5]\\
a_{25} &= \alpha_3\beta_1+\alpha_3\beta_2+\alpha_3\alpha_4\beta_2+\beta_1\beta_2+\alpha_3\beta_1\beta_2+\alpha_3\alpha_4\beta_3+\alpha_3\beta_1\beta_3+\alpha_3\beta_2\beta_3\\
&\quad +\alpha_4\beta_2\beta_3+\beta_1\beta_2\beta_3+\beta_1\beta_4+\beta_2\beta_4+\alpha_4\beta_2\beta_4+\beta_1\beta_2\beta_4+\alpha_4\beta_3\beta_4+\beta_1\beta_3\beta_4
\end{aligned}$$

$$+ \beta_2\beta_3\beta_4 + \beta_1\beta_5 + \beta_2\beta_5 + \alpha_4\beta_2\beta_5 + \beta_1\beta_2\beta_5 + \alpha_4\beta_3\beta_5 + \beta_1\beta_3\beta_5 + \beta_2\beta_3\beta_5$$
$$+ \{\beta_2 + \beta_2\beta_3 + \beta_4 + \beta_2\beta_4 + \beta_3\beta_4 + \beta_5 + (-\alpha_3 + \beta_2 + \beta_3)\beta_5 + \alpha_1[\beta_1\beta_2$$
$$+ \alpha_3(\beta_1 + \beta_2) + \beta_1\beta_4 + \beta_2\beta_4 + (\beta_1 + \beta_2)\beta_5 + \alpha_4(\beta_2 + \beta_4 + \beta_5)]$$
$$+ \alpha_2[\alpha_3(1 + \alpha_4 + \beta_1 + \beta_3) + \beta_4 + \beta_5 + \beta_3(\beta_4 + \beta_5) + \alpha_4(\beta_3 + \beta_4 + \beta_5)$$
$$+ \beta_1(1 + \beta_3 + \beta_4 + \beta_5)] + x_1 + \alpha_2^2 x_2 + \alpha_1^2 x_3\}x_5 + \{\alpha_1(\beta_2 + \beta_4 + \beta_5)$$
$$+ \alpha_2[1 + \beta_3 + \beta_4 + \beta_5 + \alpha_1(\alpha_3 + \alpha_4 + \beta_1 + \beta_4 + \beta_5)]\}x_5^2 + \alpha_1\alpha_2 x_5^3$$

$$\begin{aligned}
a_{15} = {} & \beta_2[\beta_1 + (\alpha_4 + \beta_1)\beta_3](\alpha_3 + \beta_4 + \beta_5) + \{\beta_2\beta_4 + \beta_2\beta_3\beta_4 - \alpha_3\beta_5 + \beta_2\beta_5 \\
& - \alpha_3\beta_2\beta_5 - \alpha_3\beta_3\beta_5 + \beta_2\beta_3\beta_5 + \alpha_2[\beta_1 + (\alpha_4 + \beta_1)\beta_3](\alpha_3 + \beta_4 + \beta_5) \\
& + (1 + \beta_2 + \beta_3)x_1 + \alpha_2^2(1 + \alpha_4 + \beta_1 + \beta_3)x_2 + \alpha_1^2(\alpha_4 + \beta_1 + \beta_2)x_3 \\
& + \alpha_1[(\alpha_4 + \beta_1)\beta_2(\beta_4 + \beta_5) + \alpha_3(\beta_1\beta_2 - \alpha_4\beta_5) + \alpha_4 x_1 + x_3]\}x_5 \\
& + \{\alpha_2[\beta_4 + \beta_3\beta_4 + \beta_5 - \alpha_3\beta_5 + \beta_3\beta_5 + x_1 + \alpha_2 x_2] + \alpha_1\{-\alpha_3\beta_5 + \beta_2(\beta_4 + \beta_5) \\
& + \alpha_2[\alpha_3\beta_1 + (\alpha_4 + \beta_1)(\beta_4 + \beta_5)] + x_1 + \alpha_2^2 x_2\} + \alpha_1^2(1 + \alpha_2)x_3\}x_5^2 \\
& + \alpha_1\alpha_2(\beta_4 + \beta_5)x_5^3 \\
a_{05} = {} & x_5\{\alpha_2^2\beta_1 x_2 - \alpha_3\beta_5[\beta_2 + \alpha_2 x_5][1 + \beta_3 + \alpha_1(\alpha_4 + x_5)] + \beta_2\{x_1[1 + \beta_3 \\
& + \alpha_1(\alpha_4 + x_5)] + \alpha_1 x_3[1 + \alpha_1(\alpha_4 + \beta_1 + x_5)]\} + \alpha_2\{\alpha_2 x_2[x_5 + (\alpha_4 + \beta_1 + x_5) \\
& (\beta_3 + \alpha_1 x_5)] + x_5\{x_1[1 + \beta_3 + \alpha_1(\alpha_4 + x_5)] + \alpha_1 x_3[1 + \alpha_1(\alpha_4 + \beta_1 + x_5)]\}\}\}
\end{aligned}$$

We write the Routh Hurwitz matrix as:

$$\begin{pmatrix}
a_{45} & a_{25} & a_{05} & 0 & 0 \\
1 & a_{35} & a_{14} & 0 & 0 \\
0 & a_{45} & a_{25} & a_{05} & 0 \\
0 & 1 & a_{35} & a_{15} & 0 \\
0 & 0 & a_{45} & a_{25} & a_{05}
\end{pmatrix}$$

The minors of the Routh Hurwitz matrix are given by the formulas:

$$\Delta_1 = a_{45} > 0,$$

$$\Delta_2 = \begin{vmatrix} a_{45} & a_{25} \\ 1 & a_{35} \end{vmatrix} = a_{35}a_{45} - a_{25} > 0,$$

$$\Delta_3 = \begin{vmatrix} a_{45} & a_{25} & a_{05} \\ 1 & a_{35} & a_{15} \\ 0 & a_{45} & a_{25} \end{vmatrix} = a_{25}a_{35}a_{45} + a_{05}a_{44} - R_1R_4^2 - R_2^2 = R_2\Delta_2,$$

$$\Delta_4 = \begin{vmatrix} R_4 & R_2 & R_0 & 0 \\ 1 & R_3 & R_1 & 0 \\ 0 & R_4 & R_2 & R_0 \\ 0 & 1 & R_3 & R_1 \end{vmatrix} =$$

$$=\mathrm{R}_1 R_2 R_3 R_4 + 2R_0 R_1 R_4 - R_0 R_3^2 R_4 - R_1^2 R_4^2 - R_1 R_2^2 - R_0^2 + R_0 R_2 R_3,$$

$$\Delta_5 = \begin{vmatrix} R_4 & R_2 & R_0 & 0 & 0 \\ 1 & R_3 & R_1 & 0 & 0 \\ 0 & R_4 & R_2 & R_0 & 0 \\ 0 & 1 & R_3 & R_1 & 0 \\ 0 & 0 & R_4 & R_2 & R_0 \end{vmatrix} = R_0 \Delta_4.$$

Considering the complexity of the equation, we conduct a numerical analysis of the roots.

5.4 Global stability analysis for S^+

Theorem 5.5. *The positive equilibrium point S^+ is globally asymptotically stable in the open positive orthant $\mathbb{R}_+^5$ if $a_3 = 0$.*

Proof. Let us consider an arbitrary solution $x_i(t)$, $i = \overline{1,5}$, of system (3), with initial conditions $\varphi_i : (-\infty, 0] \to (0, \infty)$. From Theorem 5.1 it follows that $x_i(t) > 0$, for any $t > 0$ and $i = \overline{1,5}$.

Knowing that $x - 1 - \ln x \geq 0$, for any $x > 0$, the Lyapunov function is constructed as follows.:

$$L(t) = L_1(t) + L_2(t) + L_3(t) + L_4(t) + L_5(t), \tag{9}$$

where

$$L_k(t) = x_k(t) - x_k^+ - x_k^+ \ln \frac{x_k(t)}{x_k^+} \ , \ k = \overline{1,5} \ , \tag{10}$$

and we denote:

$$E_k(t) = 1 - \frac{x_k(t)}{x_k^+} + \ln \frac{x_k(t)}{x_k^+} \leq 0 \quad , \quad \tilde{E}_k(t) = 1 - \frac{x_k^+}{x_k(t)} + \ln \frac{x_k^+}{x_k(t)} \leq 0 \quad , \quad k = \overline{1,5}.$$

With the notations previously described and considering the fact that S^+ is an equilibrium point of system with (3), we have:

$$L_1'(t) = \dot{x}_1 \left(1 - \frac{x_1^+}{x_1}\right) = \left(1 - \frac{x_1^+}{x_1}\right) [\gamma_1 - x_1 x_5 + \alpha_3 x_4 + x_3 - (\alpha_4 + \beta_1) x_1]$$

$$= \left(1 - \frac{x_1^+}{x_1}\right) [x_1^+ x_5^+ - \alpha_3 x_4^+ - x_3^+ + (\alpha_4 + \beta_1) x_1^+ - x_1 x_5 + \alpha_3 x_4 + x_3$$

$$
\begin{aligned}
&- (\alpha_4+\beta_1)x_1] \\
&= \left(1-\frac{x_1^+}{x_1}\right)\left[x_1^+x_5^+\left(1-\frac{x_1}{x_1^+}\frac{x_5}{x_5^+}\right)-\alpha_3x_4^+\left(1-\frac{x_4}{x_4^+}\right)-x_3^+\left(1-\frac{x_3}{x_3^+}\right)\right. \\
&\left.+(\alpha_4+\beta_1)x_1^+\left(1-\frac{x_1}{x_1^+}\right)\right] \\
&= x_1^+x_5^+\left(1-\frac{x_1}{x_1^+}\frac{x_5}{x_5^+}-\frac{x_1^+}{x_1}+\frac{x_5}{x_5^+}\right)-\alpha_3x_4^+\left(1-\frac{x_4}{x_4^+}-\frac{x_1^+}{x_1}+\frac{x_1^+}{x_1}\frac{x_4}{x_4^+}\right) \\
&- x_3^+\left(1-\frac{x_3}{x_3^+}-\frac{x_1^+}{x_1}+\frac{x_1^+}{x_1}\frac{x_3}{x_3^+}\right)+(\alpha_4+\beta_1)x_1^+\left(2-\frac{x_1}{x_1^+}-\frac{x_1^+}{x_1}\right) \\
&\le x_1^+x_5^+\left(1-\frac{x_1}{x_1^+}\frac{x_5}{x_5^+}-\frac{x_1^+}{x_1}+\frac{x_5}{x_5^+}\right)-\alpha_3x_4^+\left(2+\ln\frac{x_1^+}{x_1}+\ln\frac{x_4}{x_4^+}-\frac{x_4}{x_4^+}\right. \\
&\left.-\frac{x_1^+}{x_1}\right)-x_3\left(2+\ln\frac{x_1^+}{x_1}+\ln\frac{x_3}{x_3^+}-\frac{x_1^+}{x_1}-\frac{x_3}{x_3^+}\right)+(\alpha_4+\beta_1)x_1^+\left(E_1+\tilde{E}_1\right) \\
&= x_1^+x_5^+\left(1-\frac{x_1}{x_1^+}\frac{x_5}{x_5^+}-\frac{x_1^+}{x_1}+\frac{x_5}{x_5^+}\right)-\alpha_3x_4^+\left(\tilde{E}_1+E_4\right)-x_3^+\left(\tilde{E}_1+E_3\right) \\
&+(\alpha_4+\beta_1)x_1^+\left(E_1+\tilde{E}_1\right).
\end{aligned}
$$

Additionally, we obtain:

$$
\begin{aligned}
L_2'(t) = \dot{x}_2\left(1-\frac{x_2^+}{x_2}\right) &= \left(1-\frac{x_2^+}{x_2}\right)(\gamma_2-\alpha_2x_2x_5-\beta_2x_2) \\
&= \left(1-\frac{x_2^+}{x_2}\right)(\alpha_2x_2^+x_5^++\beta_2x_2^+-\alpha_2x_2x_5-\beta_2x_2) \\
&= \left(1-\frac{x_2^+}{x_2}\right)\left[\alpha_2x_2^+x_5^+\left(1-\frac{x_2}{x_2^+}\frac{x_5}{x_5^+}\right)+\beta_2x_2^+\left(1-\frac{x_2}{x_2^+}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \alpha_2x_2^+x_5^+\left(1-\frac{x_2}{x_2^+}\frac{x_5}{x_5^+}-\frac{x_2^+}{x_2}+\frac{x_5}{x_5^+}\right)+\beta_2x_2^+\left(2-\frac{x_2^+}{x_2}-\frac{x_2}{x_2^+}\right) \\
&= \alpha_2x_2^+x_5^+\left(1-\frac{x_2}{x_2^+}\frac{x_5}{x_5^+}-\frac{x_2^+}{x_2}+\frac{x_5}{x_5^+}\right)+\beta_2x_2^+\left(E_2+\tilde{E}_2\right).
\end{aligned}
$$

Moreover, we get:

$$
\begin{aligned}
L_3'(t) = \dot{x}_3\left(1-\frac{x_3^+}{x_3}\right) &= \left(1-\frac{x_3^+}{x_3}\right)(\alpha_4x_1-x_3-\alpha_1x_3x_5-\beta_3x_3) \\
&= \left(1-\frac{x_3^+}{x_3}\right)\left[\alpha_4(x_1-x_1^+)-(1+\beta_3)(x_3-x_3^+)-\alpha_1(x_3x_5-x_3^+x_5^+)\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \left(1-\frac{x_3^+}{x_3}\right)\left[\alpha_4 x_1^+\left(\frac{x_1}{x_1^+}-1\right) - x_3^+(1+\beta_3)\left(\frac{x_3}{x_3^+}-1\right) - \alpha_1 x_3^+ x_5^+\right.\\
&\left.\left(\frac{x_3}{x_3^+}\frac{x_5}{x_5^+}-1\right)\right] = \alpha_4 x_1^+\left(\frac{x_1}{x_1^+}-1-\frac{x_3^+}{x_3}\cdot\frac{x_1}{x_1^+}+\frac{x_3^+}{x_3}\right) + x_3^+(1+\beta_3)\\
&\left(2-\frac{x_3}{x_3^+}-\frac{x_3^+}{x_3}\right) + \alpha_1 x_3^+ x_5^+\left(1-\frac{x_3^+}{x_3}-\frac{x_3}{x_3^+}\frac{x_5}{x_5^+}+\frac{x_5}{x_5^+}\right)\\
&\le \alpha_4 x_1^+\left(-2-\ln\frac{x_3^+}{x_3}-\ln\frac{x_1}{x_1^+}+\frac{x_3^+}{x_3}+\frac{x_1}{x_1^+}\right) + x_3^+(1+\beta_3)\left(2-\frac{x_3}{x_3^+}-\frac{x_3^+}{x_3}\right)\\
&+ \alpha_1 x_3^+ x_5^+\left(1-\frac{x_3^+}{x_3}-\frac{x_3}{x_3^+}\frac{x_5}{x_5^+}+\frac{x_5}{x_5^+}\right)\\
&= \alpha_4 x_1^+(-E_1-\tilde{E}_3) + (1+\beta_3)x_3^+(E_3+\tilde{E}_3)\\
&+ \alpha_1 x_3^+ x_5^+\left(1-\frac{x_3^+}{x_3}-\frac{x_3}{x_3^+}\frac{x_5}{x_5^+}+\frac{x_5}{x_5^+}\right),
\end{aligned}
$$

and lastly:

$$
\begin{aligned}
L_4'(t) &= \dot{x}_4\left(1-\frac{x_4^+}{x_4}\right) = \left(1-\frac{x_4^+}{x_4}\right)(x_1x_5+\alpha_2x_2x_5+\alpha_1x_3x_5-\alpha_3x_4-\beta_4x_4)\\
&= \left(1-\frac{x_4^+}{x_4}\right)\left[x_1^+x_5^+\left(\frac{x_1x_5}{x_1^+x_5^+}-1\right)+\alpha_2x_2^+x_5^+\left(\frac{x_2x_5}{x_2^+x_5^+}-1\right)\right.\\
&\left.+\alpha_1x_3^+x_5^+\left(\frac{x_3x_5}{x_3^+x_5^+}-1\right)-(\alpha_3+\beta_4)x_4^+\left(\frac{x_4}{x_4^+}-1\right)\right]\\
&= x_1^+x_5^+\left(\frac{x_1x_5}{x_1^+x_5^+}-1-\frac{x_4^+x_1x_5}{x_4x_1^+x_5^+}+\frac{x_4^+}{x_4}\right)\\
&+ \alpha_2x_2^+x_5^+\left(\frac{x_2x_5}{x_2^+x_5^+}-1-\frac{x_4^+x_2x_5}{x_4x_2^+x_5^+}+\frac{x_4^+}{x_4}\right)\\
&+ \alpha_1x_3^+x_5^+\left(\frac{x_3x_5}{x_3^+x_5^+}-1-\frac{x_4^+x_3x_5}{x_4x_3^+x_5^+}+\frac{x_4^+}{x_4}\right)\\
&+ (\alpha_3+\beta_4)x_4^+\left(2-\frac{x_4}{x_4^+}-\frac{x_4^+}{x_4}\right)\\
&\le x_1^+x_5^+\left(\frac{x_1x_5}{x_1^+x_5^+}-2-\ln\frac{x_4^+}{x_4}-\ln\frac{x_1}{x_1^+}-\ln\frac{x_5}{x_5^+}+\frac{x_4^+}{x_4}\right)\\
&+ \alpha_2x_2^+x_5^+\left(\frac{x_2x_5}{x_2^+x_5^+}-2-\ln\frac{x_4^+}{x_4}-\ln\frac{x_2}{x_2^+}-\ln\frac{x_5}{x_5^+}+\frac{x_4^+}{x_4}\right)\\
&+ \alpha_1x_3^+x_5^+\left(\frac{x_3x_5}{x_3^+x_5^+}-2-\ln\frac{x_4^+}{x_4}-\ln\frac{x_3}{x_3^+}-\ln\frac{x_5}{x_5^+}+\frac{x_4^+}{x_4}\right)\\
&+ (\alpha_3+\beta_4)x_4^+(E_4+\tilde{E}_4).
\end{aligned}
$$

$$
\begin{aligned}
L_5'(t) &= \dot{x}_5\left(1-\frac{x_5^+}{x_5}\right)=\left(1-\frac{x_5^+}{x_5}\right)(x_4-\beta_5 x_5)\\
&=\left(1-\frac{x_5^+}{x_5}\right)(x_4-x_4^+-\beta_5 x_5+\beta_5 x_5^+)\\
&=\beta_5 x_5^+\left(2-\frac{x_5^+}{x_5}-\frac{x_5}{x_5^+}\right)+x_4^+\left(\frac{x_4}{x_4^+}-1-\frac{x_5^+x_4}{x_5x_4^+}+\frac{x_5^+}{x_5}\right)\\
&\le x_4^+\left(2-\frac{x_5^+}{x_5}-\frac{x_5}{x_5^+}\right)+x_4^+\left(\frac{x_4}{x_4^+}-2+\ln\frac{x_5}{x_5^+}-\ln\frac{x_4}{x_4^+}+\frac{x_5^+}{x_5}\right)\\
&=x_4^+\left(-\frac{x_5}{x_5^+}+\frac{x_4}{x_4^+}+\ln\frac{x_5}{x_5^+}-\ln\frac{x_4}{x_4^+}\right)\\
&=x_4^+(E_4-E_5)
\end{aligned}
$$

The following results are also obtained by considering the algebraic relations that the coordinates of the equilibrium point S^+ satisfies:

$$
\begin{aligned}
&L_1'(t)+L_2'(t)+L_3'(t)+L_4'(t)+L_5'(t)\le\\
&\quad\le x_1^+x_5^+\Big(1-\frac{x_1^+}{x_1}+\frac{x_5}{x_5^+}-2+\ln\frac{x_1^+}{x_1}-\ln\frac{x_4^+}{x_4}-\ln\frac{x_5}{x_5^+}+\frac{x_4^+}{x_4}\Big)\\
&\quad+\alpha_2x_2^+x_5^+\left(1-\frac{x_2^+}{x_2}+\frac{x_5}{x_5^+}-2-\ln\frac{x_4^+}{x_4}+\ln\frac{x_2^+}{x_2}-\ln\frac{x_5}{x_5^+}+\frac{x_4^+}{x_4}\right)\\
&\quad+\alpha_1x_3^+x_5^+\left(1-\frac{x_3^+}{x_3}+\frac{x_5}{x_5^+}-2-\ln\frac{x_4^+}{x_4}+\ln\frac{x_3^+}{x_3}-\ln\frac{x_5}{x_5^+}+\frac{x_4^+}{x_4}\right)\\
&\quad-\alpha_3x_4^+(\tilde{E}_1+E_4)-x_3^+(\tilde{E}_1+E_3)+(\alpha_4+\beta_1)x_1^+(E_1+\tilde{E}_1)\\
&\quad+\beta_2x_2^+(\tilde{E}_2+E_2)-\alpha_4x_1^+(E_1+\tilde{E}_3)+(1+\beta_3)x_3^+(E_3+\tilde{E}_3)\\
&\quad+(\alpha_3+\beta_4)x_4^+(E_4+\tilde{E}_4)\\
&\quad=x_1^+x_5^+(\tilde{E}_1-E_5-\tilde{E}_4)+\alpha_2x_2^+x_5^+(\tilde{E}_2-E_5-\tilde{E}_4)\\
&\quad+\alpha_1x_3^+x_5^+(\tilde{E}_3-E_5-\tilde{E}_4)-\alpha_3x_4^+(\tilde{E}_1+E_4)-x_3^+(\tilde{E}_1+E_3)\\
&\quad+(\alpha_4+\beta_1)x_1^+(E_1+\tilde{E}_1)+\beta_2x_2^+(\tilde{E}_2+E_2)-\alpha_4x_1^+(E_1+\tilde{E}_3)\\
&\quad+(1+\beta_3)x_3^+(E_3+\tilde{E}_3)+(\alpha_3+\beta_4)x_4^+(E_4+\tilde{E}_4)\\
&\quad=\tilde{E}_1\left[x_1^+x_5^+-\alpha_3x_4^+-x_3^++(\alpha_4+\beta_1)x_1^+\right]+\beta_1x_1^+E_1\\
&\quad+(\alpha_2x_2^+x_5^++\beta_2x_2^+)\tilde{E}_2+\beta_2x_2^+E_2\\
&\quad+\tilde{E}_3[\alpha_1x_3^+x_5^+-\alpha_4x_1^++(1+\beta_3)x_3^+]+\beta_3x_3^+E_3\\
&\quad+\tilde{E}_4[-x_1^+x_5^+-\alpha_2x_2^+x_5^+-\alpha_1x_3^+x_5^++(\alpha_3+\beta_4)x_4^+]+\beta_4x_4^+E_4\\
&\quad+E_5[-x_1^+x_5^+-\alpha_2x_2^+x_5^+-\alpha_1x_3^+x_5^+]
\end{aligned}
$$

$$= \gamma_1 \tilde{E}_1 + \gamma_2 \tilde{E}_2 + \beta_1 x_1^+ E_1 + \beta_2 x_2^+ E_2 + \beta_3 x_3^+ E_3 + \beta_4 x_4^+ E_4 - (\alpha_3 + \beta_4) x_4^+ E_5.$$

$\square$

6 Conclusions

In this paper we present a five-dimensional mathematical model that provides a new approach to studying the labour market, with a focus on unemployment levels, migration, fixed-term contractors, full-time employment, and the number of available vacancies. We analyse the positivity and boundedness of the solutions and discuss the existence of equilibrium points using the basic reproduction number. Local and global stability properties of the model are also demonstrated.

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